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PROJECTIVE GEOMETRY

WITH APPLICATIONS TO ENGINEERING

BY

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ILLUSTRATED

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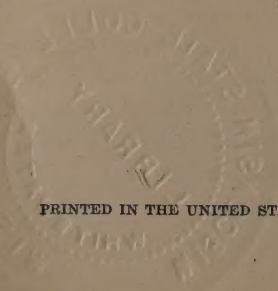

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PREFACE

In most American universities the course in descriptive geometry is purely a course in drawing. On the other hand, writers on technical mechanics frequently assume their readers have a knowledge of some of the fundamental theorems of projective geometry. It therefore seems there is a good opportunity for the mathematical departments in the technical colleges to offer an elective course in projective geometry which emphasizes the technical applications. This volume is the result of such a course which the writer gave in the University of Michigan during the first semester of the past year.

Most of our text books on projective geometry are written from the point of view of the pure mathematician and consequently it is hoped that this book will also prove useful in pointing out to students of pure mathematics some of the beautiful applications of projective geometry to mechanics. A course, which does not call attention to some of the applications, gives a one-sided and imperfect view of the subject.

No attempt has been made to give references. The authors most frequently consulted in working up the course were Cremona, Veblen and Young, Emch, Föppl, Mohr, Ritter, Culmann, and Koenigs.

I wish to express my obligation to Professors A. Ziwet, V. Snyder, and W. Denton for having read and criticized the manuscript and to Professor M. Orbeck for drawing the figures, also to Mr. D. Kazarinoff for reading the proof.

PETER FIELD

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March, 1923

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CHAPTER I

Definitions and Fundamental Notions

1. *Notation.* The elements to be dealt with are points, lines, and planes. A knowledge of such properties of these elements as are given in a course in elementary geometry is assumed. Capital letters A, B, C, $\cdots$ will be used to designate points, small letters a, b, c $\cdots$ to designate lines, and Greek letters α, β, γ $\cdots$ to designate planes. Obvious combinations of this notation will also be used, such as AB for the line joining A and B, ABC for the plane determined by the points ABC, ab for the point of intersection of two coplanar lines a and b, $\alpha\beta$ for the line of intersection of the planes α and β, etc. In cases where a symbol is needed to represent a scalar, small letters will generally be used.

2. *Metrical and Descriptive Properties.* Properties of a geometrical figure may be classified as either metrical or descriptive. The former involve the idea of measurement, such as the length of a line or the magnitude of an angle; the latter involve the idea of position, such as concurrence of lines or collinearity of points. The course in elementary analytical geometry is almost entirely devoted to metrical problems. Projective Geometry concerns itself principally with the study of descriptive properties of figures, although it will be found that the applications to engineering are mostly metrical.

3. *Projection and Section.* To project from a point O a plane figure composed of points and straight lines, lines are drawn from O to the different points, and planes are

passed through O and the different lines of the figure. The new figure is composed of lines and planes and is called the projection of the given figure from O. Its intersection with a plane which does not pass through O is a figure composed of points and lines. The second operation is known as section. Thus the points A and B and the line AB project from O into the lines OA, OB and the plane AOB. The section of this figure by a plane β gives the points A', B' and the line $A'B'$. Points collinear with O, as A and A', are called corresponding points and lines joining corresponding points such as AB and $A'B'$ are called corresponding lines, O is called the centre of perspective or centre of projection. If by any number of the processes of projection and section it is possible to pass from the figure $ABC\cdots$ to the figure $A'B'C'\cdots$, the figures are said to be projective. This will be written

$$ABC\cdots \barwedge A'B'C'\cdots,$$

read $ABC\cdots$ is projective with $A'B'C'\cdots$. Two figures which are sections of the same projection or projections of the same section are said to be in perspective or in perspective position. Two figures in perspective are also projective but the converse need not be true, but it will be found that if two plane figures are projective they can be so placed as to be in perspective position. Thus the points A and B can always be projected into any other two points A' and B' but they will not be in perspective position unless the lines AB and $A'B'$ are coplanar. In the latter case the point of intersection of AA' and BB' is the centre of perspective. In the former case, draw a line g through A' which is coplanar with AB and project A and B on g, from any point on AA', into A' and B''; $A'B''$ and $A'B'$ are in perspective.

4. *The Elements at Infinity.* In Figure 1 let O be the centre of perspective, α the plane $ABCD$, and β the plane $DCEF$. Draw the planes $OGEF$ parallel to α and $OABG$ parallel to β. There is in general a point and line in the plane β which corresponds to a point and line in the plane α and conversely, but the plane $OGAB$ cuts α in a line AB which is exceptional; similarly the plane $OGEF$ cuts β in a line FE which is exceptional. To avoid these exceptions, it will be said that the line AB in α corresponds to the line at infinity in β, that the line FE in β corresponds to the line at infinity in α, and that every plane has a line at infinity. Parallel planes will be said to intersect in a line at infinity. Every line will be said to have a point at infinity and two parallel lines will be said to intersect in a point at infinity. By this means the exceptional elements are disposed of and it can be stated that to every point and line in the plane α there corresponds a point and line in β and vice versa. For O as centre of projection, the line AB is the vanishing line for the plane β and the line FE the vanishing line for the plane α.

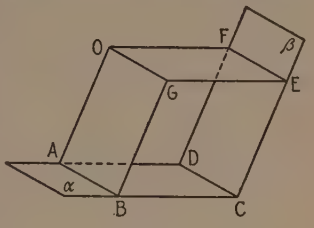

FIGURE 1

5. *The Analytical Relationship.* If after any number of projections and sections a figure in the plane α is projected into another in the plane β, there is established a relationship between the points in the two planes. Consider first the case of two figures in perspective; in Figure 2 let (x, y) and (x', y') be rectangular axes in the planes α and β respectively, so chosen that the x and x' axes coincide, let the

point O whose coordinates referred to the rectangular (x, y, z) axes are $(0, -b, c)$ be the centre of projection. Any line through O cuts α and β in the corresponding points A and A'. Draw TA' parallel to x and RA' perpendicular to α. Then if (x', y') are the coordinates of A' referred to the x', y' axes, $x' = TA'$, $y' = O_1T$, and if (x, y) are the

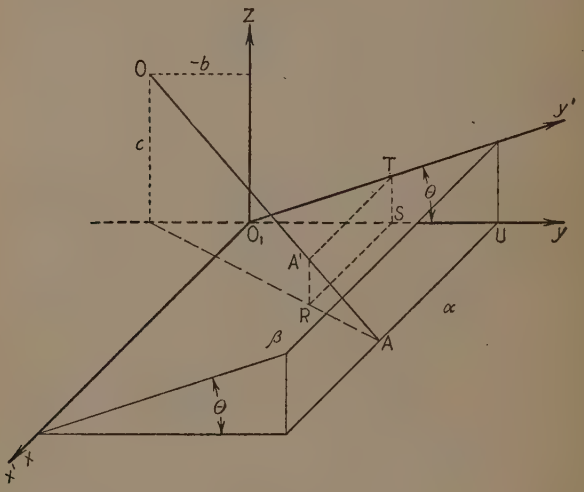

FIGURE 2

coordinates of A referred to the x, y axes, $x = UA$, $y = O_1U$. The equations of the line OA are

$$\frac{X - x}{x} = \frac{Y - y}{y + b} = \frac{Z}{-c},$$

X, Y and Z being the variables. The coordinates of A' referred to the (x, y, z) axes are found by finding the inter-

section of this line with the plane $Z = Y \tan\theta$. They are

$$Y = \frac{cy}{(y+b)\tan\theta + c}, \qquad Z = \frac{cy\tan\theta}{(y+b)\tan\theta + c},$$
$$X = \frac{x(b\tan\theta + c)}{(y+b)\tan\theta + c}.$$

Hence the coordinates of A' referred to (x', y') as axes are

$$x' = X = \frac{x(b\tan\theta + c)}{(y+b)\tan\theta + c},$$
$$y' = \frac{Y}{\cos\theta} = \frac{cy}{(y+b)\sin\theta + c\cos\theta}.$$

Now suppose that (x, y) and (x', y') are rectangular axes chosen anywhere in their respective planes. In the expressions just written down x will be replaced by an expression of the form $x\cos\varphi + y\sin\varphi - h$, y by $x\sin\varphi - y\cos\varphi - k$, x' by $x'\cos\psi + y'\sin\psi - h'$, y' by $x'\sin\psi - y'\cos\psi - k'$. The equations giving the relationship between the coordinates in the two planes become

$$x'\cos\psi + y'\sin\psi - h' = \frac{(b\tan\theta + c)(x\cos\varphi + y\sin\varphi - h)}{(x\sin\varphi - y\cos\varphi - k + b)\tan\theta + c}, \quad (1)$$

$$x'\sin\psi - y'\cos\psi - k' = \frac{c(x\sin\varphi - y\cos\varphi - k)\sec\theta}{(x\sin\varphi - y\cos\varphi - k + b)\tan\theta + c}. \quad (1)$$

These equations solved for x' and y' give expressions of the form

$$x' = \frac{l_1x + m_1y + n_1}{l_3x + m_3y + n_3}, \qquad y' = \frac{l_2x + m_2y + n_2}{l_3x + m_3y + n_3}, \quad (2)$$

or solved for x and y, expressions of similar form, as

$$x = \frac{l_1'x' + m_1'y' + n_1'}{l_3'x' + m_3'y' + n_3'}, \qquad y = \frac{l_2'x' + m_2'y' + n_2'}{l_3'x' + m_3'y' + n_3'}. \quad (2')$$

A new centre of projection can now be chosen and the points in β projected into points in a plane γ. Calling (x'', y'') the coordinates of the point in γ corresponding to (x', y'), it follows from (2′) that

$$x' = \frac{l_1''x'' + m_1''y'' + n_1''}{l_3''x'' + m_3''y'' + n_3''}, \qquad y' = \frac{l_2''x'' + m_2''y'' + n_2''}{l_3''x'' + m_3''y'' + n_3''},$$

and these values substituted in 2′ give x and y in terms of x'' and y''; x and y are each expressed as the quotient of two linear expressions in x'' and y'' with the same denominator in each. As this process can obviously be repeated, it follows that if one plane figure has been obtained from another by any number of projections and sections, the coordinates of corresponding points are connected by equations of the form given in (2) or (2′).

The lines $l_1x + m_1y + n_1 = 0$, $l_2x + m_2y + n_2 = 0$, and $l_3x + m_3y + n_3 = 0$ in the xy plane correspond to the lines $x' = 0$, $y' = 0$ and the line at infinity in the $x'y'$ plane, while the lines $l_1'x' + m_1'y' + n_1' = 0$, $l_2'x' + m_2'y' + n_2' = 0$, and $l_3'x' + m_3'y' + n_3' = 0$ in the $x'y'$ plane correspond to $x = 0$, $y = 0$ and the line at infinity in the xy plane.

6. *Special Choice of Axes.* By suitably choosing the axes of reference in each of the two planes the equations (2) may be much simplified. Choose the line in the xy plane which corresponds to points at infinity in the $x'y'$ plane as the line $y = 0$, and the line in the $x'y'$ plane which corresponds to points at infinity in the xy plane as the

line $y' = 0$. Equations (2) now become

$$x' = \frac{l_1 x + m_1 y + n_1}{y}, \qquad y' = \frac{n_2}{y}.$$

Now let x, y, z be a reference system and place the xy and $x'y'$ planes as indicated in Figure 3; i.e., with x' parallel

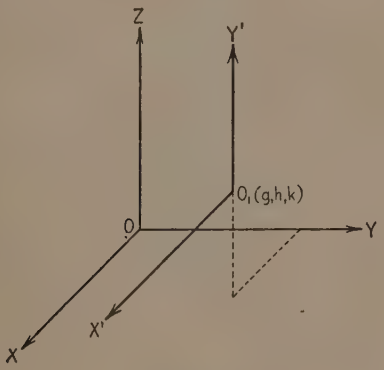

FIGURE 3

to x and y' parallel to z and with O_1 at the point (g, h, k). The coordinates of corresponding points referred to x, y, z will be $(x, y, 0)$ and

$$\left(g + \frac{l_1 x + m_1 y + n_1}{y}, \quad h, \quad k + \frac{n_2}{y}\right).$$

It is possible by properly choosing g, h and k to place the figures in the two planes so that they are in perspective position. This requires that the line joining $(x, y, 0)$ and its correspondent shall pass through a fixed point. This point will have to be in the xz plane at a height k above the x axis. The line joining $(x, y, 0)$ and its corresponding

point has the equations

$$\frac{X-x}{g+\dfrac{l_1x+m_1y+n_1}{y}-x}=\frac{Y-y}{h-y}=\frac{Z}{k+\dfrac{n_2}{y}}.$$

As $Y=0$, when $Z=k$,

$$\frac{-y}{h-y}=\frac{k}{k+\dfrac{n_2}{y}}=\frac{-y}{-y-\dfrac{n_2}{k}}.$$

This condition is satisfied, independent of y, if $hk=-n_2$. The value of X for the point in which the line pierces the xz plane is

$$\begin{aligned}X&=x+\left[g+\frac{l_1x+m_1y+n_1}{y}-x\right]\frac{y}{y-h}\\&=\frac{(l_1-h)x+(g+m_1)y+n_1}{y-h}.\end{aligned}$$

Its value is independent of x and y if $h=l_1$ and

$$\frac{n_1}{g+m_1}=-h \quad \text{or} \quad g=-m_1-\frac{n_1}{h}.$$

The line joining corresponding points will therefore pass through a fixed point if

$$g=-m_1-\frac{n_1}{l_1}, \qquad h=l_1, \qquad k=-\frac{n_2}{l_1}$$

and the coordinates of the fixed point, i.e., the centre of perspective, are $\left(-\dfrac{n_1}{l_1}, 0, -\dfrac{n_2}{l_1}\right)$. Hence the *Theorem:* If two plane figures are projective it is possible to place them so that they are in perspective position.

7. *The Geometric Forms.* A range is a figure composed

of points lying on a straight line. The line is called the base or bearer of the range.

A flat or plane pencil is a figure composed of lines lying in the same plane and passing through a common point, called the centre of the pencil. The lines are the rays of the pencil.

An axial pencil is a pencil composed of planes through a given line.

A sheaf or bundle of planes or lines is a figure composed of all the planes or lines which pass through a given point.

A plane of points or lines is a figure composed of all the points or lines in a plane.

The first three are called *onedimensional* and the last two, *twodimensional* forms. The names are justified by the fact that the onedimensional forms represent cases where there is one degree of freedom or cases where a given element is determined by one parameter while the last two cases have two degrees of freedom, it requires two parameters to fix an element.

8. *The Principle of Duality.* In the geometry of space there are certain statements which can be obtained from others by interchanging the words point and plane.

As examples:

1. A line is determined by two points.
1′. A line is determined by two planes.
2. A point is determined by three planes.
2′. A plane is determined by three points.
3. A point is determined by a line and a plane.
3′. A plane is determined by a line and a point.

In the geometry of plane figures a similar situation holds if point and line are interchanged. Thus:

4. A line is determined by two points.

4′. A point is determined by two lines.
5. Three points determine a triangle.
5′. Three lines determine a triangle.

Theorems related in this way are said to be *dual* or *correlative* to one another. It will be found that the duality relation also holds for the proofs and consequently from a given theorem and its proof it is possible to state and prove the dual.

This duality idea also enters in the cartesian coordinate system. The equation $ux + vy - 1 = 0$ represents points on a line if u and v are given fixed values and x and y are allowed to vary, but if x and y are fixed while u and v vary, it represents a family of lines through the point (x, y); so the equation $ux + vy - 1 = 0$ may be interpreted either as the points on a certain line or as the lines through a certain point.

CHAPTER II

Plane Homology

9. *Theorem:* If the figure $ABC\cdots$ in the plane α and the figure $A'B'C'\cdots$ in the plane α' are in perspective position, corresponding lines intersect on the line $\alpha\alpha'$.

Proof: Any pair of corresponding lines as AB and $A'B'$ lie in the same plane as by supposition AA' and BB' pass through the centre of perspective; hence they intersect. As AB lies in α and $A'B'$ in α', the point of intersection will lie on $\alpha\alpha'$.

Conversely: If the figure $ABC\cdots$ in the plane α and the figure $A'B'C'\cdots$ in the plane α' are such that the lines AB and $A'B'$, etc., intersect on $\alpha\alpha'$, then the two figures are in perspective position.

Proof: As AB and $A'B'$ are coplanar, AA' and BB' intersect. Similarly for AA' and CC', BB' and CC', etc. Therefore the lines must have a common point as they do not all lie in the same plane.

10. *Theorem:* If two triangles ABC and $A'B'C'$ in the same plane are so situated that AB and $A'B'$, BC and $B'C'$, AC and $A'C'$ intersect in collinear points, then AA', BB', and CC' are concurrent.

Proof: Through the line containing the intersections of the corresponding sides pass a plane β and from any point O' outside the two planes project $A'B'C'$ on β into $A''B''C''$. The new triangle is in perspective position with respect to both ABC and $A'B'C'$. Let O be the centre of perspective of ABC and $A''B''C''$. The lines $OA''A$ and $O'A'A''$ lie in the same plane, hence OO' and AA' intersect. By the

same argument OO' intersects BB' and CC'. As OO' is not in the plane of ABC and $A'B'C'$, the point of inter-

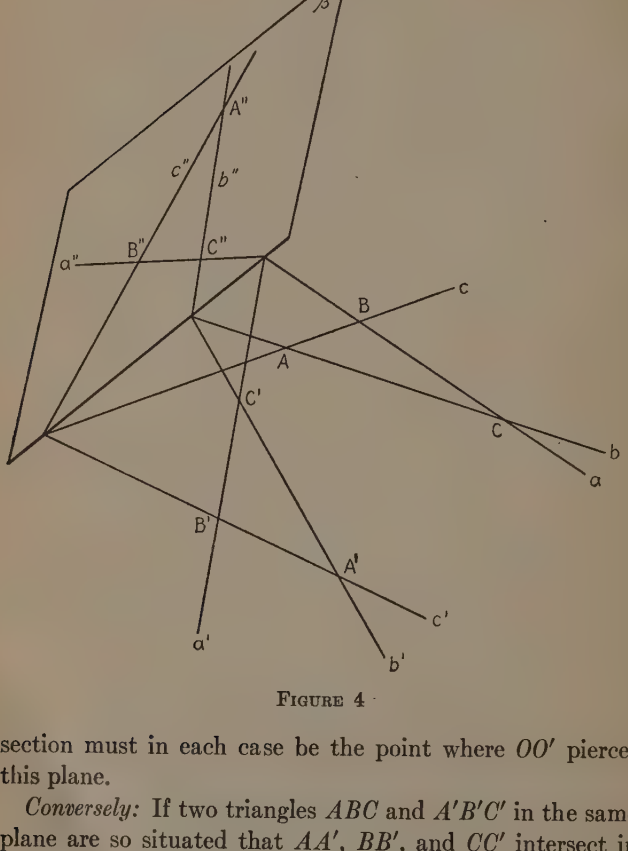

FIGURE 4

section must in each case be the point where OO' pierces this plane.

Conversely: If two triangles ABC and $A'B'C'$ in the same plane are so situated that AA', BB', and CC' intersect in a common point P, then AB and $A'B'$, BC and $B'C'$, AC and $A'C'$ intersect in collinear points.

Proof: Draw through P a line OO' not in the plane of the given triangles and project ABC from O and $A'B'C'$ from O'. As OO' and AA' both pass through P, the points A, A' and O, O' are coplanar and consequently OA and $O'A'$ intersect in some point A''; similarly OB and $O'B'$ intersect in B'', and OC and $O'C'$ intersect in a point C''. The triangle $A''B''C''$ is in perspective with both the given triangles and AB and $A'B'$ must both pass through the point where $A''B''$ pierces the plane of the given triangles. Similarly for the other sides.

11. *Definition:* Two figures in the same plane, so related that corresponding points lie on lines which pass through a point O and corresponding lines intersect on a line s, are said to be in plane homology. The point O is the centre of homology and the line s is the axis of homology. Two homological figures may be regarded as projections from different points of the same figure. The homology may be determined in various ways as (a) by s, O and a pair of corresponding points which must be collinear with O, (b) by s and two pairs of corresponding points which must be such that the two corresponding lines intersect on s, (c) by two triangles such that AA', BB', and CC' are concurrent.

12. *Exercise:*

1. Given O, s and a pair of corresponding points, find the line in one figure which corresponds to the line at infinity in the other.
2. If O is regarded as a point of one figure, what is its corresponding point in the other?
3. If s is regarded as a line of one figure, what is its corresponding line in the other?
4. Given O, s and a pair of corresponding points,

find the figure corresponding to a given conic.
(*b*) Solve the problem if s is the line at infinity.
(*c*) Solve the problem if O is at infinity.

5. Show by means of projection and section that four points in a plane α, no three being collinear, can be projected into any four such points in a plane β.

6. In Ex. 5, if α and β coincide, show that plane homologies can be used in place of projection and section. *Hint:* Let A, B, C, D and A', B', C', D' be the two sets of four points. First show that A', B', C' can be projected into A, B, C; the new position of D' being D''. Then use AB as axis and C as centre of perspective and project D'' into the point where CD'' intersects AD. Call this point D'''. Now use A as centre and BC as axis of perspective.

13. The subject of plane homology or plane perspective can also be introduced in connection with Art. 6. Everything in Art. 6 will hold irrespective of the angle between the xy plane and the parallel planes xz and $x'y'$. Hence if the xz and $x'y'$ planes be turned about their lines of intersection with the xy plane until y' and z coincide in direction and sense with y, the centre of perspective will fall at $\left(-\frac{n_1}{l_1}, -\frac{n_2}{l_1}, o\right)$, the point O_1 will fall at $\left(-m_1 - \frac{n_1}{l_1}, l_1 - \frac{n_2}{l_1}, o\right)$ and the line of intersection of the xy and $x'y'$ planes will remain at $y = l_1$. The situation will be as shown in Figure 5.

Referring points in both planes to xy as a common reference system, the coordinates of corresponding points are

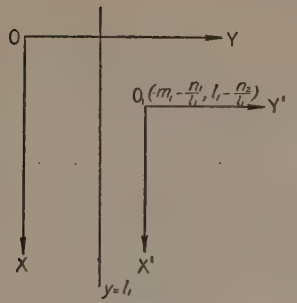

FIGURE 5

(x, y) and

$$\left(-m_1-\frac{n_1}{l_1}+\frac{l_1x+m_1y+n_1}{y},\ l_1-\frac{n_2}{l_1}+\frac{n_2}{y}\right)$$

and it is easily shown that corresponding points are collinear with the point $\left(-\frac{n_1}{l_1}, -\frac{n_2}{l_1}\right)$ and that corresponding lines intersect on $y = l_1$; in other words $\left(-\frac{n_1}{l_1}, -\frac{n_2}{l_1}\right)$ is the centre, $y = l_1$ is the axis of homology, and the (xy) and $(x'y')$ points are corresponding points in a plane homology.

CHAPTER III

Linkages

14. *Definition:* A plane linkage is defined by Koenigs as "a combination of plates or plane figures subject to remain in one and the same plane, among which a certain number are connected to each other by hinges or pivots perpendicular to the common plane."

The mechanisms which arise in connection with the construction of machinery are ordinarily linkages with one degree of freedom as for instance in connecting rod motion where one end of the connecting rod moves along a straight line while the other moves along the arc of a circle. If two points $P(x, y)$ and $P'(x', y')$ are so related that when one is given the other is determined, then x' and y' are functions of x and y but the relation is a projective one only in case this relationship is of the form given in equations (2). Hence only such linkages as satisfy this condition have a place in a course in Projective Geometry. The following articles are intended to clarify the meaning of projective transformations by carrying out the process by mechanical means.

15. *The Group of Motions.* The group of motions includes the operations of translation, rotation, and a combination of the two. The equations (2) take the special form

$$\begin{aligned} x' &= x\cos\theta - y\sin\theta + h, \\ y' &= x\sin\theta + y\cos\theta + k. \end{aligned} \tag{3}$$

When the planes of (x, y) and (x', y') and also the axes

are made coincident (Fig. 6), every point (x, y) is brought

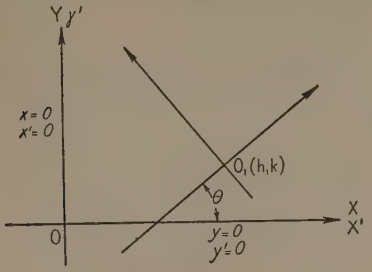

FIGURE 6

into coincidence with its correspondent (x', y') by giving the XOY plane a translation OO_1 and then turning it through an angle θ about the point O_1.

16. A translation can be realized mechanically by the use of Kempe's Translator shown in Figure 7.

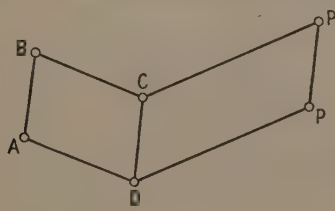

FIGURE 7

The linkage consists of the two parallelograms $ABCD$ and $CDPP'$. When A and B are fixed and P describes any figure, P' will describe a figure which can be obtained from the one described by P by giving the latter a translation AB.

17. *Sylvester's Pantograph.* Sylvester's Pantograph is shown in Figure 8. It consists of a parallelogram $OABC$

having O fixed and the two similar triangles ABP and $CP'B$. It will be shown that when P moves, the position of P' can be obtained by turning OP through a constant angle and making the distance OP' a constant multiple of OP. As a particular case this multiple may be one

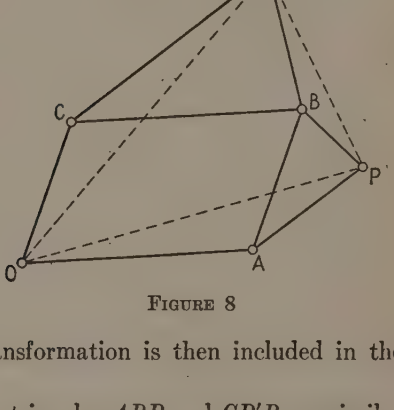

FIGURE 8

and the transformation is then included in the group of motions.

Proof: As triangles ABP and $CP'B$ are similar

$$AB : AP :: CP' : CB \quad \text{or} \quad OC : CP' :: AP : OA.$$

Hence triangles OCP' and OAP are similar as they have $\angle OCP' = \angle OAP$ and the sides adjacent to these angles are proportional. Therefore $OP : OP' = OC : AP$, a constant. The triangle PBP' is similar to the triangles OCP' and PAO for

$$\begin{aligned} \angle PBP' &= 2\pi - \angle CBA - \angle P'BC - \angle ABP \\ &= 2\pi - \angle CBA - \pi + \angle BCP' \\ &= \pi - \angle CBA + \angle BCP' \\ &= \pi + \angle BCP' - \pi + \angle OCB = \angle OCP'. \end{aligned}$$

It is given that

$$AB : BP = CP' : BP' \quad \text{or that} \quad OC : CP' = BP : BP'.$$

The triangles PBP' and OCP' are therefore similar and

$$\angle\, OP'P = \angle\, CP'B \text{ a constant},$$
$$\angle\, OPP' = \angle\, BPA = \angle\, CBP' \text{ a constant},$$

$\angle\, POP' = \pi - \angle\, CP'B - \angle\, CBP' = \angle\, BCP'$ a constant, which proves the theorem.

It may be noted that of the three points O, P, and P' it does not matter which one is fixed, the correspondence between the other two is of the same nature in each case.

18. *Peaucellier's Inversor.* Let there be given a rhombus $AP'BP$ of side b and two links of length a connecting A and B with a fixed point O. Let p be the length of the per-

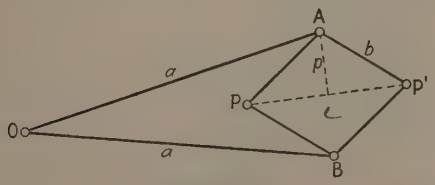

FIGURE 9

pendicular from A on PP'. For all positions of P the points O, P, and P' are collinear and it follows at once from the figure that

$$\left(\frac{OP + OP'}{2}\right)^2 + p^2 = a^2,$$
$$\left(\frac{OP' - OP}{2}\right)^2 + p^2 = b^2.$$

Hence

$$OP \cdot OP' = a^2 - b^2. \tag{4}$$

P and P' are therefore inverse with respect to a circle whose centre is O and whose radius is $\sqrt{a^2 - b^2}$. If an additional link of length k is introduced in Figure 9, one end of the link being attached to P and the other fixed at a point on the perpendicular bisector of OP, there results the linkage known as Peaucellier's Inversor. As P moves along the arc of a circle through O and of radius k, the polar equation of P with O as pole and the line from O to the centre of the circle as initial line is

$$OP = 2k \cos \theta.$$

This value substituted in equation (4) gives

$$OP' \cos \theta = \frac{a^2 - b^2}{2k}.$$

Therefore P' describes a line perpendicular to the line from O to the centre of the circle $OP = 2k \cos \theta$ and at a distance $(a^2 - b^2)/2k$ from O.

The relation between P and P' is not a projective one but nevertheless it is of interest as it furnishes a linkage for describing a straight line.

19. *The Inversor Pantograph.* Two linkages such as given in Figure 9 connected as shown in Figure 10 give a

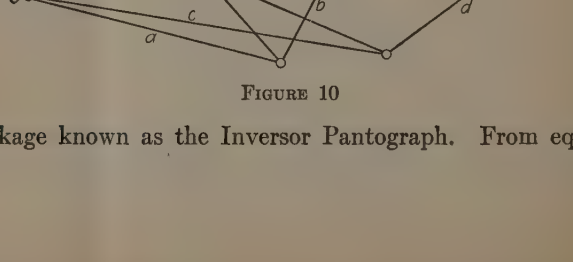

FIGURE 10

linkage known as the Inversor Pantograph. From equa-

tion (4)

$$OP \cdot OQ = a^2 - b^2, \qquad OP' \cdot OQ = c^2 - d^2, \text{ by division,}$$
$$\frac{OP}{OP'} = \frac{a^2 - b^2}{c^2 - d^2}.$$

The Inversor Pantograph therefore accomplishes mechanically the same transformation as is accomplished by a plane perspective taking O as centre of perspective, s the axis of perspective as the line at infinity and a pair of corresponding points P and P' such that

$$\frac{OP}{OP'} = \frac{a^2 - b^2}{c^2 - d^2}.$$

With cartesian coordinates having O as origin and the (x, y) and (x', y') axes coincident, the corresponding transformation is

$$x' = \frac{c^2 - d^2}{a^2 - b^2} x, \qquad y' = \frac{c^2 - d^2}{a^2 - b^2} y.$$

20. *The Stork's Beak.* The stork's beak shown in Figure 11 accomplishes the same transformation as the linkage of the preceding section. It consists of a parallelogram $ADPB$ with the sides AD and AB extended to O and P', points so chosen that

$$OD : DP = OA : AP'.$$

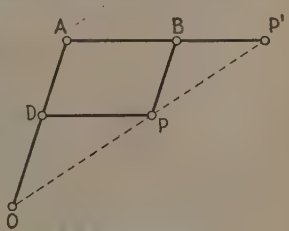

FIGURE 11

When O is fixed the points O, P and P' will remain collinear and the distances OP and OP' will remain in a constant ratio. This linkage is in common usage under the name of Scheiner's Pantograph.

21. An *Affine* Transformation or an *affinity* is a per-

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spective having the centre of perspective at infinity. It will first be obtained mechanically.

Take two similar rhombs $AEPF$ and $ADCB$ with equal angles at A and insert links $FB = ED$ of such length that

$$\angle DAE = \angle BAF = \frac{\pi}{2}.$$

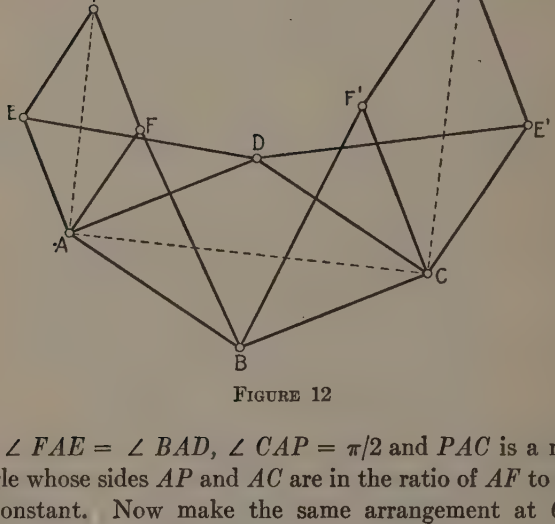

FIGURE 12

As $\angle FAE = \angle BAD$, $\angle CAP = \pi/2$ and PAC is a right angle whose sides AP and AC are in the ratio of AF to AD, a constant. Now make the same arrangement at C as at A and consider the relation between P and P' when A and C are constrained either by pins or by Peaucellier Inversors so that they move along the same straight line.

Take the x axis as the line along which A and C move and the y axis at right angles to this line at any point. Let (x, y) and (x', y') be the coordinates of P and P' referred

to these axes. Then

$$x' = x + AC = x + y\frac{AD}{AF},$$

$$y' = CP' = \frac{CF'}{AD}AC = \frac{CF'}{AD}\frac{AD}{AF}y = \frac{CF'}{AF}y$$

or

$$x' = x + my, \qquad y' = ny, \tag{5}$$

where m and n are constants whose values are the ratios $AD : AF$ and $CF' : AF$. The relation between P and P' is therefore projective, Points on the line along which A and C move go into themselves. The X axis must therefore be the axis of perspective. The line at infinity goes into itself. It therefore passes through O, the centre of perspective, and O is at infinity. To find its direction note that to the point $P \equiv (0, 1)$ corresponds the point $P' \equiv (m, n)$.

Exercise: Use equations (5) to prove that corresponding points are collinear with O and that corresponding lines intersect on the x axis.

22. Ritter's Perspectograph carries out the general perspective transformation mechanically. Suppose given a figure in the plane α which is to be projected on β from O. Through O draw γ, a plane parallel to β; the parallel lines b and c being the intersections of α with β and γ. Drop perpendiculars from O and P' on c and b. Lay off on c, $RO_1 = RO$ and on b, $QP_1' = QP'$. The line O_1P_1' passes through P. Let P'' be the point where P' will fall when the plane β is turned about b till it is brought into coincidence with α.

Then what is needed is the following:

Two rods with hinges at R and O_1 (Figure 14) and having

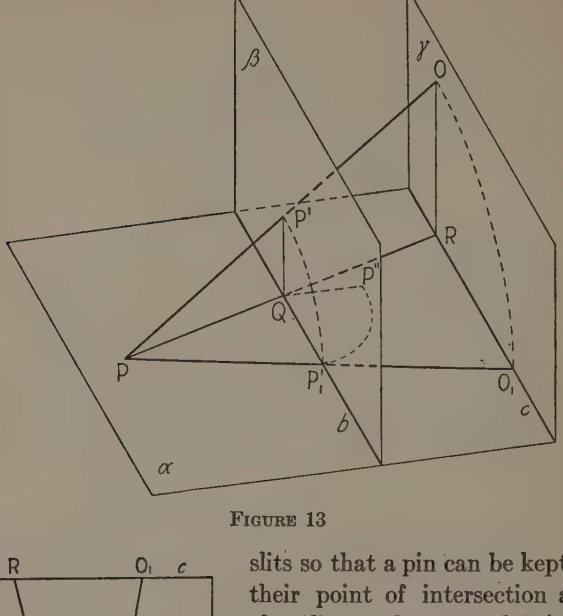

FIGURE 13

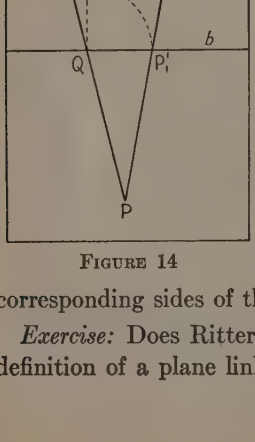

FIGURE 14

slits so that a pin can be kept at their point of intersection and also slits so that Q and P_1' can travel along b. To locate P'' use the two rhombs $ABCD$ and $AFPE$ with the connecting links ED and BF as shown in Figure 12, placing A at Q and C at P_1'. Only when the two rhombs are equal does P'' trace the perspective figure, in other cases it traces the perspective figure magnified in the ratio of the length of corresponding sides of the rhombs.

Exercise: Does Ritter's Perspectograph satisfy Koenig's definition of a plane linkage?

CHAPTER IV

The Complete Quadrangle and Quadrilateral

23. Four coplanar points, no three of which are collinear, and the six lines joining them in pairs form a complete quadrangle. The points are the vertices, the lines are the sides, lines which do not pass through the same vertices are opposite. The three pairs of opposite sides intersect in three points which form the diagonal triangle.

Theorem: If five pairs of sides of two complete quadrangles intersect in collinear points, the sixth pair will intersect on the same line.

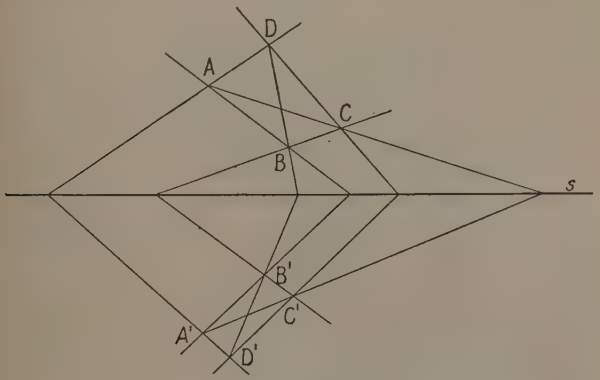

Figure 15

Given the two complete quadrangles $ABCD$ and $A'B'C'D'$ with AB and $A'B'$, BC and $B'C'$, CD and $C'D'$, AD and $A'D'$, BD and $B'D'$ intersecting on s, to prove that the remaining pair AC and $A'C'$ also intersect on s.

Proof: Corresponding sides of the triangles ABD and $A'B'D'$ intersect on s, therefore (Art. 10) they are in plane perspective. The same is true of triangles BCD and $B'C'D'$. The axis of perspective is s and the centre of perspective is the point of intersection of BB' and DD' in both cases. Hence AA', BB' and CC' are concurrent and consequently triangles ABC and $A'B'C'$ are in plane perspective (Art. 10). As AB and $A'B'$, BC and $B'C'$ intersect on s, s is their axis of perspective. Therefore AC and $A'C'$ intersect on s.

24. Interchanging the words point and line in paragraph 23 gives the following. Four lines no three of which are concurrent and their six points of intersection form a complete quadrilateral. The lines are the sides, the points are the vertices, vertices which do not lie on the same side are opposite. The three pairs of opposite vertices determine three lines which form a triangle, the diagonal triangle.

Theorem: If five pairs of vertices of two complete quadrilaterals lie on concurrent lines, the sixth pair will lie on a line passing through the same point.

Exercise: Write out the proof of the theorem just given and draw a figure.

25. *Theorem:* The intersections of corresponding sides of two funicular polygons lie on a straight line parallel to the line joining the two poles.

Given the plane system of forces f_1, f_2, $f_3 \cdots$ (Fig. 16), construct the two force polygons (Fig. 17) using O and O' as poles. At any point P on f_3 (Fig. 16) draw c and d parallel to the c and d in the force polygon and at Q draw c' and d' parallel to the c' and d' in the force polygon. Call R and S the intersections of dd' and cc'. To prove that the intersections of corresponding sides of the two funicular polygons lie on a line parallel to OO'.

Proof: The quadrangle $OO'DC$ in Figure 17 and the quadrangle $SRQP$ in Figure 16 have five pairs of sides parallel, therefore the sixth side SR is parallel to OO'. In the same way a line parallel to b' drawn through T intersects a line parallel to b drawn through U at a point W such that SW is parallel to OO' and hence S, W and R are collinear. As this process can be extended indefinitely the theorem is proved.

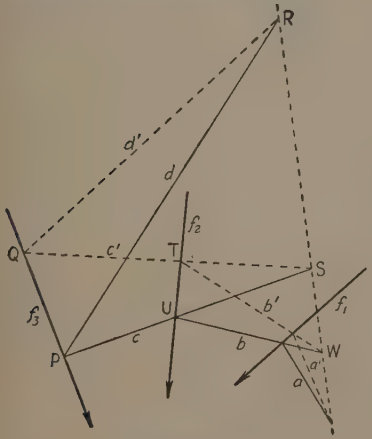

FIGURE 16

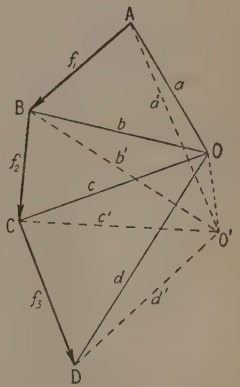

FIGURE 17

If the forces are concurrent their point of intersection is the centre of perspective and the two polygons are in plane perspective.

26. *Problem:* Given beams AB and BC with pivots at A, B and C. The resultants of the loads on AB and BC are r_1 and r_2 respectively. It is desired to find the stress on each pivot.

Offhand it may be noticed that it is a statically deter-

minate problem as there are six unknowns and the conditions for equilibrium of AB and BC each furnish three conditions. What is wanted is to construct a funicular polygon which passes through A, B and C. Figure 19 shows a force polygon with O' as pole. In Figure 18 there is drawn the corresponding funicular polygon passing through A but not through B and C. Any other funicular polygon is in perspective with the one constructed and has M as centre of perspective. Hence the points B' and C' corresponding to B and C can be found. The axis of perspective s passes through A, as A is a self-corresponding point, and the point of intersection of BC and

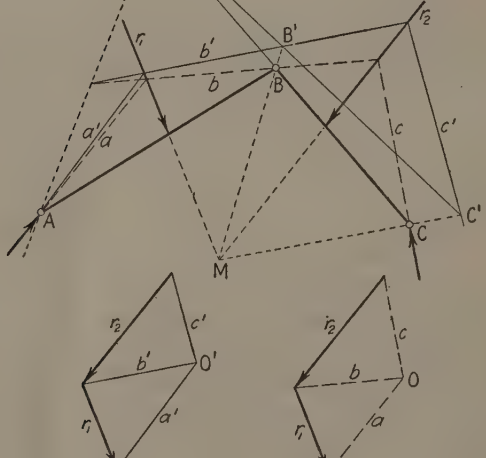

FIGURE 18

FIGURE 19 FIGURE 20

$B'C'$. Having s, the lines b, a, c can be drawn and the stresses on the pivots will be as shown in Figure 20.

27. Another class of problems in graphical statics which might have been taken up at the end of the chapter on plane homology will be introduced at this point, as the connection with the problems just treated is fairly close.

The reader is assumed to be familiar with the equation of the elastic line of a beam

$$\frac{d^2y}{dx^2} = \frac{M}{EI}.$$

Starting with this equation Mohr hit on the idea of considering the elastic line as the funicular polygon for a loading represented by the area under the bending moment curve. To prove this let ACB (Fig. 21) be the bending

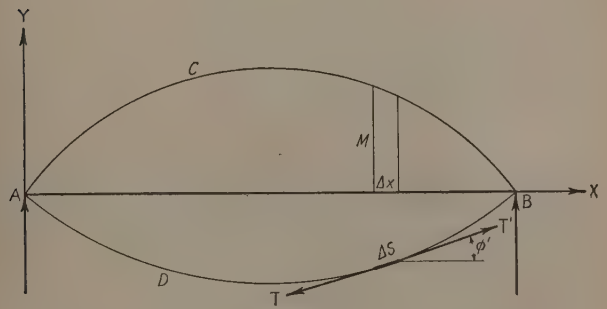

FIGURE 21

moment curve and ADB the corresponding funicular polygon, i.e., the form assumed by a string when loaded in the manner indicated. The horizontal component H of the tension in the string is a constant. Consider the equilibrium of a portion Δs of the string. Call T and T'

the tension of the string at the two ends of Δs and φ and φ' the angle which the tangent to the curve at these points makes with Ox. The load on Δs is $M\Delta x$. Then

$$T' \cos \varphi' = T \cos \varphi = H,$$
$$T' \sin \varphi' = T \sin \varphi + M\Delta x.$$

Solving the first two equations for T and T' in terms of H and substituting in the third gives

$$H \tan \varphi' = H \tan \varphi + M\Delta x,$$

or

$$H\Delta\left(\frac{dy}{dx}\right) = M\Delta x,$$

hence

$$\frac{\Delta\left(\frac{dy}{dx}\right)}{\Delta x} = \frac{M}{H}.$$

When Δx approaches zero this reduces to

$$\frac{d^2y}{dx^2} = \frac{M}{H}.$$

This is the same as the equation of the elastic line if H is taken equal to EI. In addition the boundary conditions must be the same. While the value of H can be controlled by varying the length of the string it is of no consequence as the different curves have their ordinates in a constant ratio.

28. Suppose given a beam of length $l + l_1$ (Fig. 22). Over the part AB there is a uniform load w per unit length, over the part BC there is no load. It is required to find the reactions of the three supports, at A, B, and C, the weight of the beam being neglected.

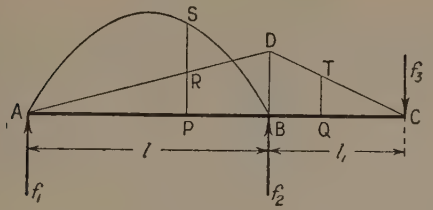

FIGURE 22

Let f_1, f_2 and f_3 be the reactions at A, B and C, directed as in Figure 22. It is evident that

$$f_1 = w\frac{l}{2} - f_3\frac{l_1}{l}, \qquad f_2 = f_3\frac{l_1 + l}{l} + \frac{wl}{2}.$$

The bending moment on a section at a distance x from A is

$$w\frac{l}{2}x - \frac{wx^2}{2} - f_3\frac{l_1}{l}x \quad \text{if} \quad x < l$$

and it is $-f_3(l_1 + l - x)$ if $x > l$. The bending moment at a point P between A and B is therefore $PS - PR$ where

$$PS = w\frac{l}{2}x - \frac{wx^2}{2} \qquad \text{and} \qquad PR = f_3\frac{l_1 x}{l}.$$

At a point Q between B and C it is $-QT = -f_3(l_1 + l - x)$. The distance BD is $f_3 l_1$ and is as yet unknown.

Following Mohr's idea, the load on the beam is represented by the area under the parabola which is known. The resultant of this load is a force f_4 directed downward at the mid-point of AB (Fig. 23). In addition there are the two loads represented by the areas of the triangles ABD and DBC which give resultants f_5 and f_6 directed

upward. Although BD is unknown the line ss which contains the resultant of f_5 and f_6 can be constructed.

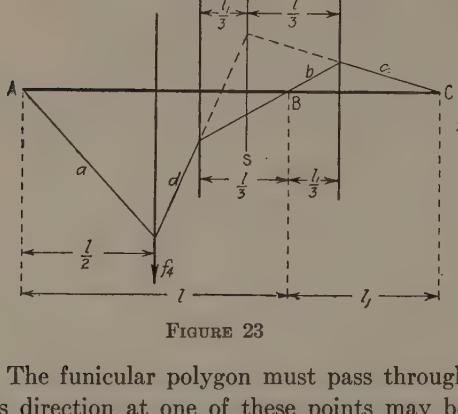

FIGURE 23

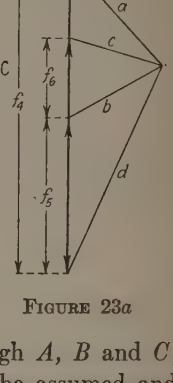

FIGURE 23a

The funicular polygon must pass through A, B and C; its direction at one of these points may be assumed and the others will then be determined. Start out by assuming the tangent to the funicular polygon at C as the line c, where c cuts f_6 draw a line through B, where b intersects f_5 draw a line d which passes through the point of intersection of c and ss. Connect A with the point where d cuts f_4 by the line a. The ratio of the unknowns f_5 and f_6 to the known f_4 is shown in Figure 23a; hence f_3 can be determined and the problem is solved.

29. The problem of the last paragraph involved no projective geometry but it seemed best to take it up before considering a slightly more complicated one which does. Figure 24 represents a beam with supports at A, B, C and D. The span between B and C carries a uniform load w per unit of length, the other two spans carry no load as the weight of the beam is neglected. It is required to find the reactions of each of the supports.

Figure 25 shows the bending moment diagram which represents the load which is to determine the funicular

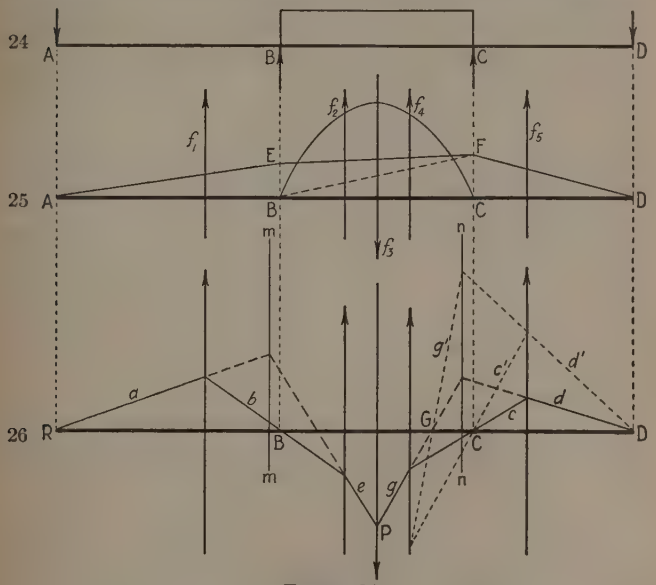

FIGURE 24
FIGURE 25
FIGURE 26

polygon, both the positive and negative being drawn above the line *ABCD*. *BE* and *CF* are unknown; the former is the product of the reaction at *A* by the length *AB*, the latter the product of the reaction at *C* by the length *CD*. The areas of the triangles *ABE*, *BFE*, *BCF* and *CDF* are negative; they represent loads directed upward. The area under the parabola has a known resultant f_3; f_1, f_2, f_4 and f_5 are the resultants of the loads represented

by the four triangles, mm and nn (Fig. 26) are the lines of action of the resultants of f_1, f_2 and f_4, f_5, respectively.

Referring to Figure 26, the line a can be assumed and b and e are then determined. The line g can not be drawn offhand. To determine g, first draw any line d' through d, at the point where it intersects f_5 draw c' through C; then draw g' through the points where d' meets nn and the point where c' meets f_4. The triangle $g'c'd'$ satisfies all but one condition—g' cuts the line of action of f_3 at a point P' instead of at the point P where f_3 meets the line e. The triangle wanted is homologous with the given triangle, has $ABCD$ as axis of perspective, the centre of perspective at infinity, and P and P' as a pair of corresponding points. The triangle gcd is therefore readily constructed. Figure 27 shows the magnitudes of the unknowns f_1, f_2, f_4 and f_5 in comparison with the known f_3. The values of f_1 and f_5 lead at once to the values of the reactions at A and D and knowing them the reactions at B and C are easily determined. In the funicular polygon the sides a, b, c and d give the direction of the elastic line at A, B, C and D excepting that the slope in each case is a multiple of the actual one due to choosing a arbitrarily.

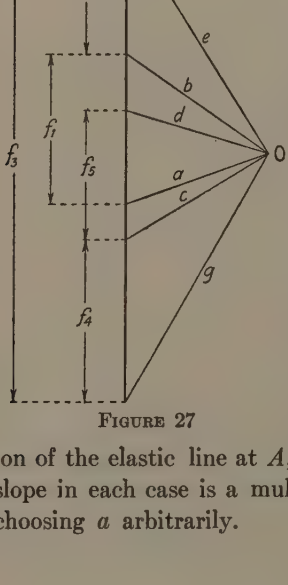

FIGURE 27

CHAPTER V

The Cross-Ratio

30. *Definition:* The cross-ratio or anharmonic ratio of four collinear points A, B, C, D is the ratio of

$$\frac{AC}{BC} \text{ to } \frac{AD}{BD},$$

and is written $(ABCD)$. It is positive if both C and D divide the segment AB internally or if they both divide it externally, otherwise negative. The cross-ratio of a pencil of four lines formed by joining A, B, C and D to any point O is the ratio

$$\frac{\sin AOC}{\sin BOC} : \frac{\sin AOD}{\sin BOD}$$

and is positive or negative according as the cross-ratio of the points A, B, C, D is positive or negative.

Theorem: The cross-ratio of four points is equal to the cross-ratio of the pencil formed by joining the points to any point O.

Given the collinear points A, B, C, D and the pencil formed by joining them to O. To prove:

$$\frac{\sin AOC}{\sin BOC} : \frac{\sin AOD}{\sin BOD} = \frac{AC}{BC} : \frac{AD}{BD} = (ABCD).$$

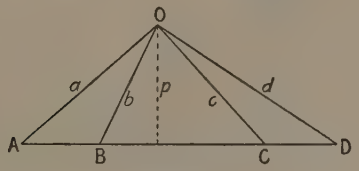

Figure 28

Let p be the length of the perpendicular from O on the line $ABCD$ and a, b, c, d the distances between O and the points A, B, C, D. The areas of the different triangles are in the ratio of the segments intercepted on the base. Also

$$\begin{aligned} 2 \text{ Area } \triangle AOC &= ac \sin AOC, \\ 2 \text{ Area } \triangle BOC &= bc \sin BOC, \\ 2 \text{ Area } \triangle AOD &= ad \sin AOD, \\ 2 \text{ Area } \triangle BOD &= bd \sin BOD. \end{aligned}$$

$$\therefore \frac{ac \sin AOC}{bc \sin BOC} : \frac{ad \sin AOD}{bd \sin BOD} = \frac{AC}{BC} : \frac{AD}{BD},$$

$$\frac{\sin AOC}{\sin BOC} : \frac{\sin AOD}{\sin BOD} = (ABCD).$$

The cross-ratio is consequently unaltered by projection and section. It is a metrical property which is also projective. It should be noted that if two segments as AC and BC are opposite in sense, the same is true of the corresponding angles AOC and BOC.

31. *Theorem:* When three of the points A, B, C, D on a line l and the value r of the cross-ratio are given, the fourth point can be determined.

Let D be the point to be determined. Project A, B, C from a point O on a line l' which is parallel to OC and suppose $ABCD$ projects into $A'B'C'D'$. Then

$$r = (ABCD) = \frac{AC}{BC} : \frac{AD}{BD} = \frac{A'C'}{B'C'} : \frac{A'D'}{B'D'}$$

$$= \frac{B'D'}{A'D'} = (A'B'C'D').$$

But when $\frac{A'D'}{B'D'} = r$, D' can be constructed by the methods of elementary geometry, hence D' is found and OD' cuts l in the required point D.

It is known (Ex. 5 and 6, Art. 12) that any four points in a plane, no three being collinear, can be projected into any other such a set of four points. If the points A, B, C, D be projected into A', B', C', D' in a second plane, any point E in the first plane corresponds to a definite point E' in the second. For the pencils A–$BCDE$ and A'–$B'C'D'E'$ have the same cross-ratio, the same is true of the pencils B–$ACDE$ and B'–$A'C'D'E'$. E' is therefore determined as the point of intersection of the two rays.

32. *Number of Cross-Ratios.* Four letters can be arranged in twenty-four different orders but the corresponding cross-ratios are not all different for

$$(ABCD) = \frac{AC}{BC} : \frac{AD}{BD} = \frac{BD}{AD} : \frac{BC}{AC} = (BADC) =$$

$$\frac{CA}{DA} : \frac{CB}{DB} = (CDAB) = \frac{DB}{CB} : \frac{DA}{CA} = (DCBA).$$

This can also be shown directly by projection, for (Fig. 29) $ABCD$ projects from E into $FBOG$, this set projects from D into $FALE$, and $FALE$ projects from O into $BADC$. Similarly for the others. The result can be stated: A set of four points is projective with the set derived by interchanging the elements in pairs.

As the elements can be interchanged in pairs without changing the value of the cross-ratio, it is possible to get all the values of the cross-ratios for different arrangements of the four elements by keeping one of them fixed and interchanging the others. This gives $(ABCD)$, $(ABDC)$,

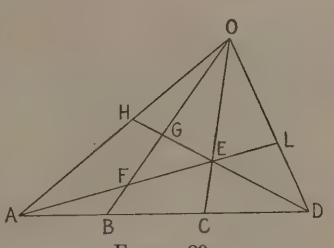

FIGURE 29

$(ACBD)$, $(ACDB)$, $(ADBC)$, and $(ADCB)$. If $(ABCD) = r$, the other five cross-ratios are readily obtained by projecting $ABCD$ into $A'B'C'D'$, D' being at infinity. Then

$$(ABCD) = (A'B'C'D') = \frac{A'C'}{B'C'} : \frac{A'D'}{B'D'} = \frac{A'C'}{B'C'} = r,$$

$$(ABDC) = (A'B'D'C') = \frac{B'C'}{A'C'} = \frac{1}{r},$$

$$(ACBD) = (A'C'B'D') = \frac{A'B'}{C'B'} = \frac{A'C' + C'B'}{C'B'} = 1 - r,$$

$$(ACDB) = (A'C'D'B') = \frac{C'B'}{A'B'} = \frac{1}{1 - r},$$

$$(ADBC) = (A'D'B'C') = \frac{A'B'}{A'C'} = \frac{A'C' + C'B'}{A'C'}$$

$$= 1 - \frac{1}{r} = \frac{r - 1}{r},$$

$$(ADCB) = (A'D'C'B') = \frac{A'C'}{A'B'} = \frac{r}{r - 1}.$$

Hence the six possible values of the cross-ratio are

$$r, \quad \frac{1}{r}, \quad 1 - r, \quad \frac{1}{1 - r}, \quad \frac{r - 1}{r}, \quad \text{and} \quad \frac{r}{r - 1}.$$

Exercise: Derive the six values of the cross-ratio just given without projecting into another set of four points having an element at infinity.

33. *Harmonic Points.* Four collinear points A, B, C, D are harmonically situated if $(ABCD) = -1$. The segment AB is then divided internally and externally in the same ratio by C and D. The six values of the cross-ratio reduce to 3; viz. -1, 2, and $\frac{1}{2}$.

Theorem: A line joining two diagonal points of a com-

plete quadrangle is divided harmonically by the other two sides.

Given the complete quadrangle E, F, G, H with A and B two diagonal points and the points of intersection of AB with the two remaining sides at C and D (Fig. 30), to prove $(ABCD) = -1$.

Proof: $ABCD$ projects from F into $GEKD$ and this set in turn projects from H into $BACD$.

Hence

$$(ABCD) = (BACD) = (ABDC)$$

or

$$r = \frac{1}{r}, \quad r = -1.$$

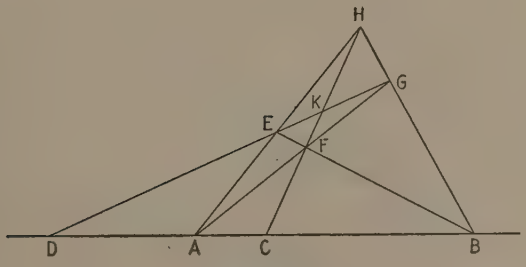

FIGURE 30

34. *Exercise:*

1. State and prove the dual of the theorem of Article 33.
2. If $(ABCD) = -1$ and $AC = CB$, where is D?
3. If $(ABCD) = -1$ and three of the points A, B, C, D are given, show how to find the fourth.
4. Show how to find the fourth ray in a harmonic pencil.

35. In a plane homology, the centre of homology O and any point M on s, the axis of homology, correspond to themselves. If A and B are points on the line OM which

correspond to $A'B'$,

$$(ABOM) = (A'B'OM),$$
$$\frac{AO}{BO} : \frac{AM}{BM} = \frac{A'O}{B'O} : \frac{A'M}{B'M},$$
$$\frac{AO \cdot BM}{BO \cdot AM} = \frac{A'O \cdot B'M}{B'O \cdot A'M},$$
$$\frac{AO \cdot A'M}{A'O \cdot AM} = \frac{BO \cdot B'M}{BM \cdot B'O},$$
$$\frac{AO}{A'O} : \frac{AM}{A'M} = \frac{BO}{B'O} : \frac{BM}{B'M},$$

i.e., $(AA'OM) = (BB'OM)$.

Moreover if C and C' are corresponding points on a line which cuts the axis of homology in Q, then AC and $A'C'$ intersect at a point N on s (Fig. 31) and hence $(AA'OM) = (CC'OQ)$. The ratio $(AA'OM)$ has therefore a constant value for a given plane homology, i.e., it is independent of the particular pair of corresponding points chosen. When its value is -1, the homology is harmonic or involutorial. The reason for the latter name will appear later.

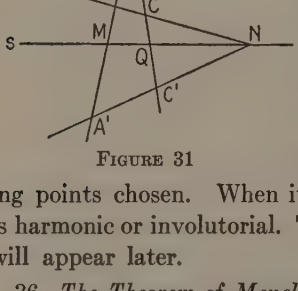

FIGURE 31

36. *The Theorem of Menelaus.* If a transversal cuts the sides of a triangle QRS in $A'B'C'$, it determines on them segments connected by the relation

$$\frac{QA' \cdot RB' \cdot SC'}{SA' \cdot QB' \cdot RC'} = 1.$$

Conversely: If the relation just stated is satisfied, A', B', C' are collinear.

Construct the complete quadrangle $SRQT$ with the diagonal points Q' and S' at infinity (Fig. 32).

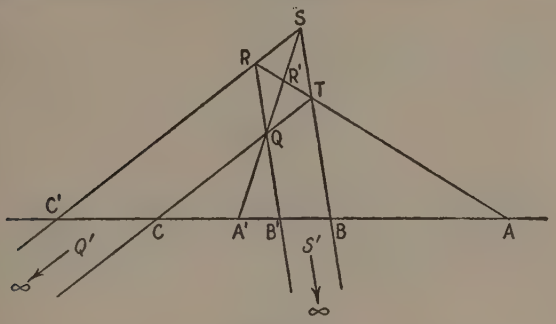

FIGURE 32

Proof: $SQR'A' \barwedge BCAA'$, $QRS'B' \barwedge CABB'$, $RSQ'C' \barwedge ABCC'$, the centre of projection being T in each case. Hence

$$(SQR'A')\,(QRS'B')\,(RSQ'C') = (BCAA')\,(CABB')\,(ABCC'),$$

i.e.,

$$\frac{SR'\cdot QA'\cdot QS'\cdot RB'\cdot RQ'\cdot SC'}{QR'\cdot SA'\cdot RS'\cdot QB'\cdot SQ'\cdot RC'} = \frac{BA\cdot CA'\cdot CB\cdot AB'\cdot AC\cdot BC'}{CA\cdot BA'\cdot AB\cdot CB'\cdot BC\cdot AC'}.$$

But $\frac{SR'}{QR'} = -1$ and $\frac{QS'}{RS'} = 1$, $\frac{RQ'}{SQ'} = 1$, so it follows that

$$\frac{QA'\cdot RB'\cdot SC'}{SA'\cdot QB'\cdot RC'} = \frac{CA'\cdot AB'\cdot BC'}{BA'\cdot CB'\cdot AC'}.$$

It remains to prove that the right-hand member of the last equation is equal to one. Notice that $ACA'B'$ projects from Q into $ATR'R$ and the latter projects from S into $ABA'C'$. Equating their cross-ratios

$$\frac{AA'\cdot CB'}{CA'\cdot AB'} = \frac{AA'\cdot BC'}{BA'\cdot AC'}.$$

Multiplying by the reciprocal of the left-hand member gives $$1 = \frac{CA' \cdot AB' \cdot BC'}{BA' \cdot AC' \cdot CB'},$$ which completes the proof.

Conversely: Given that

$$\frac{QA' \cdot RB' \cdot SC'}{SA' \cdot QB' \cdot RC'} = 1.$$

To prove that A', B', and C' are collinear.

Proof: Suppose the line $A'B'$ meets the third side in a point C'', then by the theorem just proved

$$\frac{QA' \cdot RB' \cdot SC''}{SA' \cdot QB' \cdot RC''} = 1,$$

and $\frac{SC''}{RC''}$ has a given value, but by supposition $\frac{SC'}{RC'}$ has the same value and consequently C'' and C' coincide.

37. The theorem of Menelaus can be applied to prove a theorem regarding the position of the instantaneous centres of four bars connected by pivots. Let DA, AB, BC, and CD be the four bars with pins at A, B, C, and D. Call E, F, G, and H the instantaneous centres, and ω_1, ω_2, ω_3, and ω_4 the angular velocities (Fig. 33). The theorem is: The line joining the instantaneous centres of two opposite sides passes through the point of intersection of the other two sides. Referring to the figure, EG must pass through O and FH through I.

Proof:

$$FB\omega_2 = -BG\omega_3, \qquad CG\omega_3 = -HC\omega_4.$$

Hence

$$FB \cdot CG\omega_2\omega_3 = BG \cdot HC \cdot \omega_3\omega_4,$$

or

$$\frac{FB \cdot CG}{BG \cdot HC} = \frac{\omega_4}{\omega_2}.$$

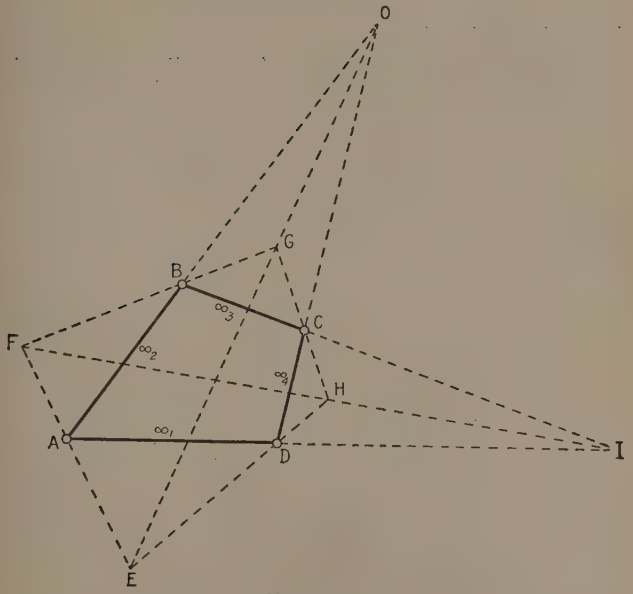

FIGURE 33

From the theorem of Menelaus

$$\frac{FI \cdot GB \cdot HC}{FB \cdot GC \cdot HI} = 1,$$

and substituting for $\frac{GB \cdot HC}{FB \cdot GC}$ its value $\frac{\omega_2}{\omega_4}$ gives

$$\frac{FI}{HI} = \frac{\omega_4}{\omega_2}.$$

Similarly, by extending AD till it meets FH in I',

$$\frac{FI'}{HI'} = \frac{\omega_4}{\omega_2},$$

hence I and I' coincide.

By similar argument it is proved that GE passes through the intersection of AB and CD.

CHAPTER VI

Ranges and Pencils

38. *Theorem:* Any three points of a range can be projected into any other three.

Given ABC on a line l and $A'B'C'$ on a line l', to prove that $A'B'C'$ can be projected into ABC.

Proof: From a point on the line AA' project $A'B'C'$ on a line through A and coplanar with $A'B'C'$ into the points $AB''C''$. The ranges ABC and $AB''C''$ are in perspective position and have the centre of perspective at the point of intersection of BB'' and CC''. As the cross-ratio is unaltered by projection it follows that any fourth point of the first range goes into a definite point of the second.

Exercise: State and prove the dual.

39. *Theorem:* Two coplanar projective ranges are both in perspective position with respect to a third range.

Given projective ranges on the lines l and l' and the corresponding points PAQ and $P'A'Q'$, P and Q' being at the point ll' (Fig. 34). A range on $P'Q$ is in perspective position with respect to both the given ranges.

Construction: Draw AA' meeting PQ' in A''.

Proof: $P'A''Q$ and PAQ are in perspective because Q is a self-corresponding point. Their centre of perspective is A'. $P'A''Q$ and $P'A'Q'$ are in perspective with A as centre of perspective. Hence the range $P'A''Q$ is in perspective position with respect to both the given ranges.

To get B', the point on l' which corresponds to B on l, draw BA' meeting l'' at B'', then will AB'' meet l' in B' the desired point; i.e., AB' and $A'B$ meet on a line l''.

But A, A' and B, B' are any pair of corresponding points so the theorem can also be stated: If $ABCD\cdots$ and

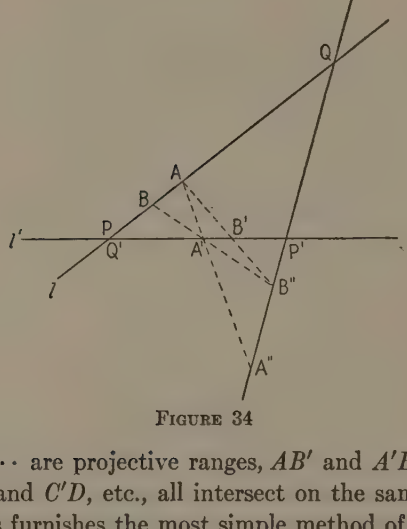

FIGURE 34

$A'B'C'D'\cdots$ are projective ranges, AB' and $A'B$, BC' and $B'C$, CD' and $C'D$, etc., all intersect on the same straight line. This furnishes the most simple method of constructing corresponding points when the ranges are not in perspective position.

Dually: If $abcd\cdots$ and $a'b'c'd'\cdots$ are corresponding rays of two projective pencils, then the pairs of points ab' and $a'b$, bc' and $b'c$, etc., lie on concurrent lines.

Exercise: Prove the theorem just stated.

40. *Theorem:* If the sides of a variable n-gon rotate about fixed points on a straight line and if $n-1$ of the vertices at the same time move along fixed straight lines, then the remaining vertex also moves along a straight line. The theorem will be proved for $n = 4$. The method of proof for the general case is the same.

Given $ABCD$ a variable quadrilateral (Fig. 35) whose vertices A, B, D move along the lines a, b, d, and whose sides AB, BC, CD, and DA turn about the collinear points H, E, F, and G, to prove the locus of C is a straight line.

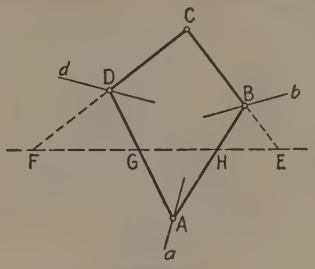

FIGURE 35

Proof: AD and DC describe perspective ranges on d, AD and BA on a, BA and CB on b, hence FDC and EBC describe projective ranges, but FE is a self-corresponding ray. Therefore the pencils are in perspective position and the locus of C is a straight line.

41. In space a force can be resolved along three lines. The magnitudes of the components can be found by the methods of descriptive geometry. In Figure 36, a, b, c

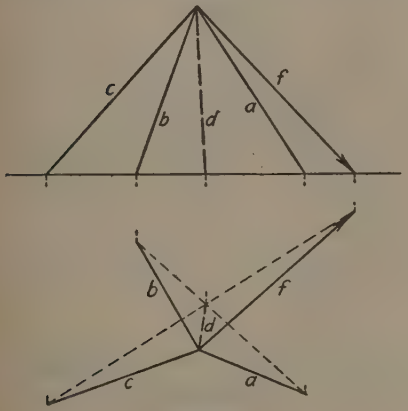

FIGURE 36

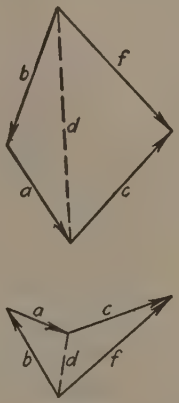

FIGURE 37

represent the horizontal and vertical projections of the three lines along which the force f is to be resolved into components f_a, f_b and f_c.

One method of solving the problem is to say that the resultant of f and $-f_c$ is equal to the resultant of f_a and f_b and must therefore be along d, the line of intersection of the plane ab and the plane fc. This being known the vertical and horizontal projections of f_a, f_b, f_c can at once be constructed as in Figure 37.

A second method makes use of the theorem of Article 40. Draw (Fig. 38) the horizontal and vertical projections of a segment of a line parallel and equal to f and at its extremities draw lines parallel to b and c. The problem is to put in a line parallel to a so that the quadrilateral, which is not plane, will close. In order to prevent confusion in what follows, points are lettered differently in their horizontal and vertical projections. Now $ABCD$ is a quadrilateral whose sides remain parallel, i.e., they all rotate about points on the line at infinity and A, B, and C move along straight lines. Therefore D describes a straight line and all that is needed is to get two positions of D; the line joining them cuts c in the point sought.

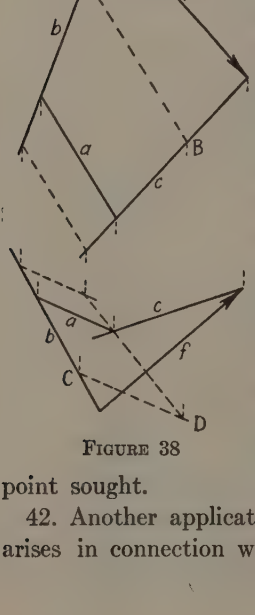

FIGURE 38

42. Another application of the theorem of Article 40 arises in connection with the construction of the stress

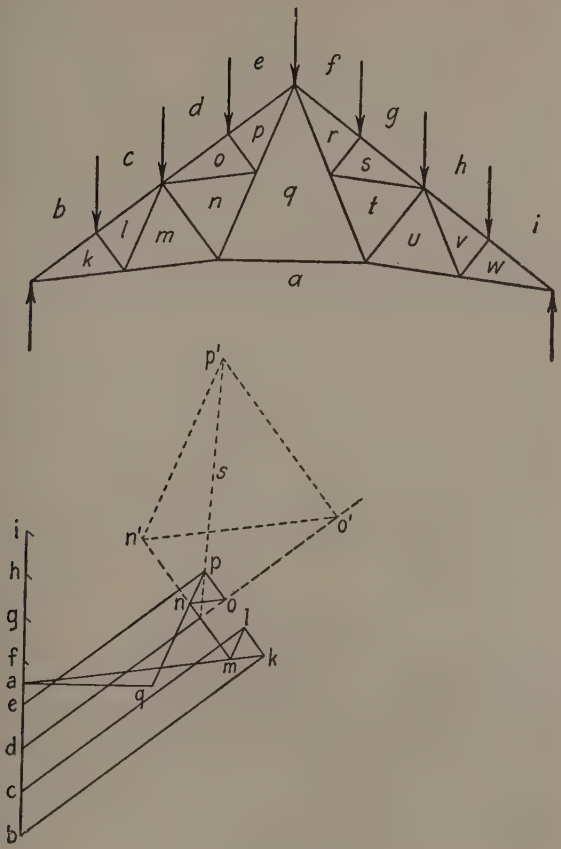

FIGURE 39
FIGURE 40

diagram (Fig. 40) for the framework shown in Figure 39. The notation is that usually employed in graphical statics.

The points l, m, k in the stress diagram are constructed without any difficulty. To determine the point o assume a point o' and then the triangle $o'p'n'$ is determined. For different positions of o' the sides of the triangle remain parallel; i.e., they rotate about points on the line at infinity, n' and o' move along fixed straight lines and accordingly (Art. 40) p' describes a straight line, the line s. The position of p must be such that ep is parallel to do'. This determines p and the triangle nop can be constructed and the stress diagram completed.

43. *Definition:* Two ranges such that distances between corresponding points are equal are called equal ranges. Two pencils such that angles between corresponding rays are equal are called equal pencils.

Theorem: The locus of points of intersection of corresponding rays of two equal pencils is a circle.

Given the two equal pencils a, b, $c\cdots$ and a', b', $c'\cdots$ with corresponding rays intersecting at A, B, $C\cdots$, to prove A, B, $C\cdots$ lie on a circle.

Proof: Angles at A, B, $C\cdots$ are equal. Therefore A, B, $C\cdots$ lie on a circle passing through the centres of the two pencils.

44. *Theorem:* A variable tangent to a circle determines on two fixed tangents two projective ranges.

Given t and t' two fixed tangents to a circle cut by a moving tangent a in A and A', to prove that A and A' generate projective ranges.

Let T, T' and Q be the points of tangency of t, t' and a.

Proof:

$$\angle\, QOA = \tfrac{1}{2} \angle\, QOT,$$
$$\angle\, A'OQ = \tfrac{1}{2} \angle\, T'OQ.$$

Adding

$$\angle\, A'OA = \tfrac{1}{2} \angle\, T'OT, \text{ a constant.}$$

Hence OA and OA' are equal pencils. The ranges generated by A and A' are projective as they are sections of equal pencils.

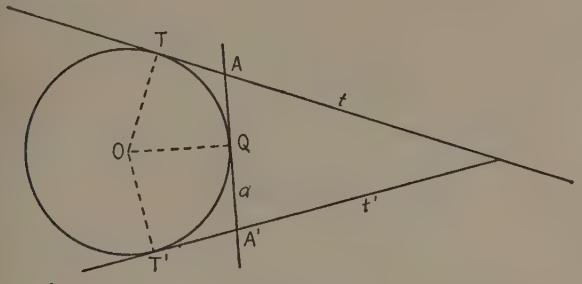

FIGURE 41

45. *Theorem:* The locus of the point of intersection of corresponding rays of two projective pencils is a conic section through the centres of the two pencils. For two perspective pencils the conic degenerates into a pair of lines.

Given the projective pencils $abc\cdots$ and $a'b'c'\cdots$ with

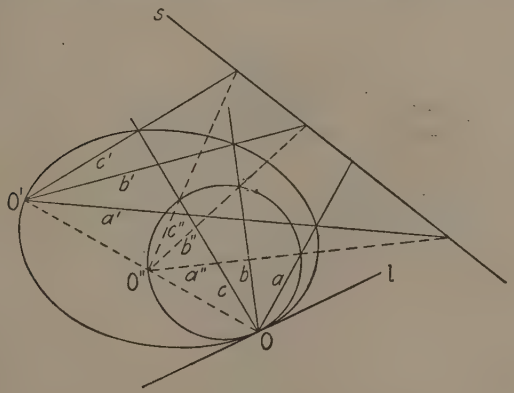

FIGURE 42

centres at O and O', to prove corresponding rays intersect on a conic section. Let l be the line in the first pencil which corresponds to $O'O$ in the second. Draw a circle tangent to l at O and cutting OO' at O''. Let a'', b'', $c''\cdots$ be a pencil at O'' which is equal to the pencil at O.

Proof: The pencils at O' and O'' are in perspective position as both are projective with the pencil at O and $O''O$ and $O'O$ is a self-corresponding ray. Therefore, corresponding rays intersect on a line s and the construction is that for obtaining the figure homologous to a circle given O the centre of perspective, s the axis of perspective and a pair of corresponding points O' and O''. If the pencils are in perspective position, the conic degenerates into two lines, their axis of perspective and the line joining their vertices.

46. *Theorem:* The lines joining corresponding points of two projective ranges envelop a conic. If they are in

FIGURE 43

perspective position the conic degenerates into a pair of points, the point of intersection of the bearers of the ranges and the centre of perspective.

Given $ABC\cdots$ and $A'B'C'\cdots$, two projective ranges on lines s and s'. Construct S the point corresponding to S' (Fig. 43) and draw a circle tangent to s at S. Draw s'' tangent to the circle from ss', also draw tangents to the circle from A, B, $C\cdots$ meeting s'' in A'', B'', $C''\cdots$.

Proof: The ranges A'', B'', $C''\cdots$ and A', B', $C'\cdots$ are not only projective but in perspective position as S' and S'' coincide. Calling O the centre of perspective of the two ranges, the construction is that for finding the figure homologous to the given circle, regarding the circle as the envelope of its tangents and being given O the centre of perspective, s the axis of perspective and the pair of corresponding lines s' and s''.

CHAPTER VII

The Hexagon

47. A hexagon is a figure composed of six points, the vertices, and six lines, the sides, which connect the successive points. If the points are traversed in the order $AB'CA'BC'$, then AB' and $A'B$, BC' and $B'C$, CA' and $A'C$ are the three pairs of opposite sides. By choosing different orders for the points 60 different hexagons may be formed having the same vertices.

Brianchon's Theorem: If a hexagon is circumscribed to a conic, the straight lines joining the three pairs of opposite vertices are concurrent.

Given the hexagon $ab'ca'bc'$ circumscribed to a conic with the opposite vertices Q and B, Q' and A, R' and S, to prove QB, $Q'A$ and $R'S$ are concurrent.

Proof: The ranges $a(b'ca'c')$ and $b(b'ca'c')$ are projective (Art. 46) and therefore the pencils A–$S'P'R'Q'$ and B–$SPRQ$ are not only projective but in perspective position as they

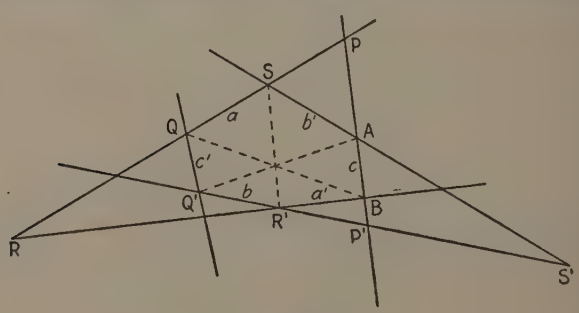

Figure 44

have a self-corresponding ray, c. Also AR' and BR intersect at R', AS' and BS intersect at S, hence $R'S$ is the axis of perspective and the corresponding rays AQ' and BQ intersect on $R'S$.

48. *Exercise:*

1. Construct a conic when five tangents are given.
2. Given five tangents to a conic, find the point of contact of one of them.
3. Given four tangents to a parabola, construct additional tangents. Hint: The fifth tangent is the line at infinity.
4. Four tangents are given to a parabola, find the direction of its axis.
5. Given three tangents to a conic and the points of contact of two of them, find the point of contact of the third.

49. *Pascal's Theorem:* The three pairs of opposite sides of a hexagon inscribed in a conic intersect in points on a straight line. This being the dual of Brianchon's Theorem, the proof is left as an exercise.

Exercise:

1. Construct a conic which passes through five given points.
2. Find the tangent to the conic at one of the five given points.

50. The hexagon and its three diagonals as a framework. Given the hexagon $AB'CA'BC'$ and its three diagonals, acted on by forces at the six vertices, the external forces being in equilibrium, the stresses in the different members are to be determined.

Let f_1, f_2, f_3 (given only in position in Figure 45) be the

sum of the external forces at the pairs of opposite vertices, A and A', B and B', C and C' respectively, then the three vectors

$$f_1 + f_2 + f_3 = 0.$$

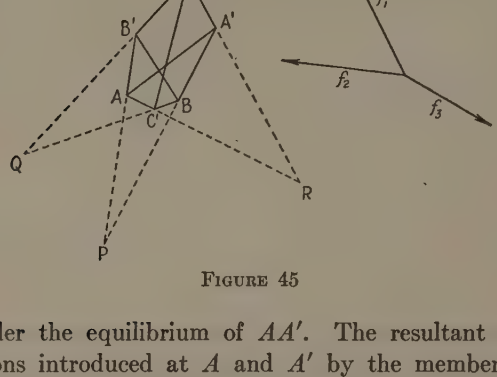

FIGURE 45

Consider the equilibrium of AA'. The resultant of the reactions introduced at A and A' by the members AB' and $A'B$ is a force f_4 through P and the resultant of the reactions of AC' and $A'C$ is a force f_5 through R. The same consideration regarding the other diagonals introduces a force f_6 through Q; P, Q, and R being the points of intersection of the pairs of opposite sides.

The next step is to get the magnitude and direction of f_4, f_5, and f_6. They pass through P, Q, and R and form a triangle whose vertices are on the lines of action of f_1, f_2, and f_3.

Construction: Let l_1, l_2, and l_3 (Fig. 46) meeting in O be the lines of action of f_1, f_2, f_3 and P, Q, and R, as before, the points of intersection of the pairs of opposite sides. Through P draw any line meeting l_1 and l_2 in A' and B', draw the line RA' meeting l_3 in C', and also draw the line

$B'C'$ meeting OQ in Q'. The triangle wanted is constructed by finding a triangle homologous with $A'B'C'$, having O as centre and PR as axis of perspective and Q' and Q as a pair of corresponding points. This gives the triangle ABC.

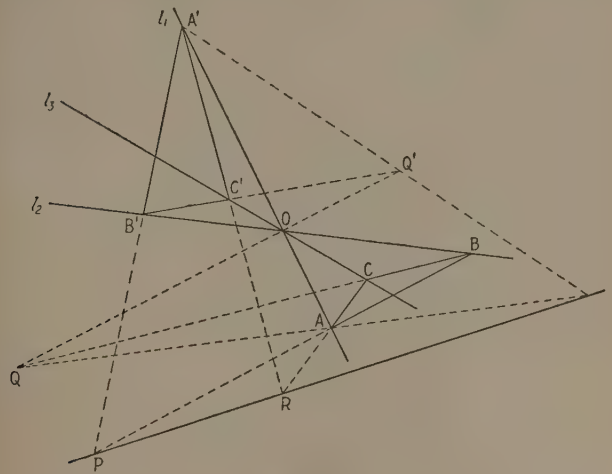

FIGURE 46

When the directions of the forces f_4, f_5 and f_6 have been found their magnitude is found as indicated in Figure 47, and the stresses in the individual sides can be determined. In case the external force at each vertex is given in place of the resultant for each pair of opposite vertices the stresses in the diagonals are also readily determined.

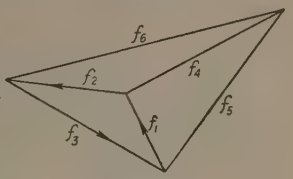

FIGURE 47

It should be noticed that in Figure 46 if Q approaches the line PR, the triangle ABC approaches a straight line and this leads to large stresses in Figure 47. In other words the general hexagon with its three diagonals forms a rigid framework but the Pascal Hexagon does not.

51. To construct a polygon whose sides shall pass through given points, all but one of whose vertices shall lie on given lines, and the angle included by the sides which meet at the one vertex which is not constrained to a line shall have a given value.

Take the case of a triangle, the procedure for the general case being the same. Let the vertices A and B be constrained to the lines a and b (Fig. 48), let U, V and W be the points through which AB, BC and CA are to pass. Draw any lines through U and V which intersect on b, and through W draw WA' so the angle VCW is of the re-

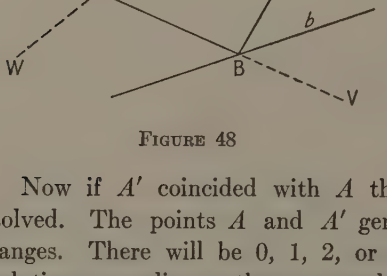

FIGURE 48

quired size. Now if A' coincided with A the problem would be solved. The points A and A' generate two projective ranges. There will be 0, 1, 2, or an infinite number of solutions according as these ranges have 0, 1, 2 or an infinite number of self-corresponding points. Or it can be put this way: the conic formed as the intersection

of the corresponding rays of the pencils at W and U may not cut a (0 solution), it may touch a (1 solution), it may intersect a (2 solutions), it may break up into two lines (an infinite number of solutions if a is one of the lines).

52. Self-corresponding points of two superposed projective ranges. Given two superposed projective ranges $ABC\cdots$ and $A'B'C'\cdots$, to find their self-corresponding points.

Find J' the point which corresponds to J the point at infinity on the line regarded as a point of the first range, find I the point which corresponds to I' the point at infinity on the line regarded as a point of the second range, and call M a self-corresponding point which is to be found. Then

$$(JIAM) = (J'I'A'M),$$

$$\frac{JA \cdot IM}{IA \cdot JM} = \frac{J'A' \cdot I'M}{I'A' \cdot J'M},$$

but $\frac{JA}{JM} = 1$ and $\frac{I'M}{I'A'} = 1$, so that

$$\frac{IM}{IA} = \frac{J'A'}{J'M} \qquad \text{or} \qquad IM \cdot J'M = IA \cdot J'A' = \pm k^2.$$

The value of k is determined by the well-known construction on a semicircle as in Figure 49 where PQ and QR are of the same length as the segments IA and $J'A'$, no regard being taken of the sense, and $k^2 = \overline{QS}^2$.

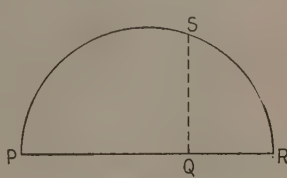

FIGURE 49

Case $IM \cdot J'M = -k^2$. The unknown point M divides the segment IJ' internally. Construct a semicircle on

IJ' as a diameter and draw a line parallel to IJ' and at a distance k, cutting the circle in F and G, perpendiculars dropped from F and G on IJ' meet the latter at M and N, both self-corresponding points. There are 2, 1 or 0 solutions according as k is less than, equal to, or greater than $\frac{IJ'}{2}$.

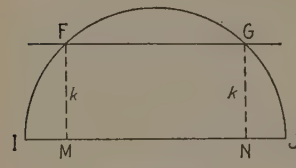

FIGURE 50

Case $IM \cdot J'M = k^2$. The unknown point divides the segment IJ' externally. Construct a circle of diameter IJ' and at any point lay off a tangent MF of length k. Draw a diameter $MIOJ'$ and lay off $ON = MO$. M and N are both self-corresponding points, placed with respect to I and J' as in the figure, and they will always be real.

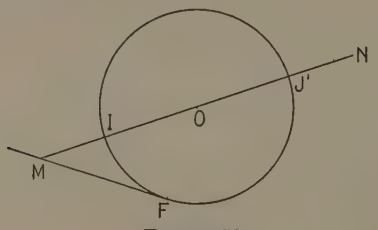

FIGURE 51

53. *Problem:* A ray of light emanating from a given point O is reflected from n given straight lines in succession; to determine the original direction which the ray must have in order that this may make with its direction after the last reflexion a given angle.

Let u_1, $u_2 \cdots$ be the successive reflecting lines, a, b, $c \cdots$ the broken path of the ray of light, φ_1, $\varphi_2 \cdots$ the angles

between the successive reflecting lines, O the source of light, and $\pi/2 - \alpha$ the angle of incidence. Let U_1 be the point symmetric to O with respect to u_1, U_2 symmetric to U_1 with respect to u_2, etc. As the angle of incidence is equal to the angle of reflexion, b passes through U_1 (Fig. 52), c through U_2, etc. Hence this is a special case of Article 51.

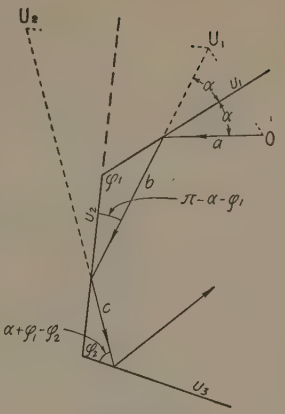

FIGURE 52

The complements of successive angles of incidence are

$$\alpha, \quad \pi - \alpha - \varphi_1, \quad \alpha + \varphi_1 - \varphi_2,$$
$$\pi - \alpha - \varphi_1 + \varphi_2 - \varphi_3, \cdots.$$

Hence the total angle which the ray of light turns through is

$$2(\alpha + \pi - \alpha - \varphi_1 + \alpha + \varphi_1 - \varphi_2 + \pi - \alpha - \varphi_1 + \varphi_2 - \varphi_3 \cdots).$$

If the number of reflecting surfaces is even, this angle is independent of α, i.e., the angle between the first incident and the final reflected ray is the same irrespective of the angle of incidence. For n an odd number, the angle is a function of α.

A practical application occurs in Bauernfeind's angle mirror or optical square in which $n = 2$, and $\varphi_1 = \pi/4$. The angle between the first incident and the last reflected ray is then

$$2\left(\pi - \frac{\pi}{4}\right) = \frac{3}{2}\pi,$$

i.e., the rays are at right angles.

CHAPTER VIII

Involution

54. In the case of two projective ranges on bearers a and b, projecting the second range on the bearer of the first from an arbitrary point O gives two superposed projective ranges. A point will in general correspond to a certain other point but this second point will depend on whether the first is regarded as a point of the first or of the second range. An analogous statement holds regarding two projective pencils. If O is chosen on the line $A'B$ (Fig. 53),

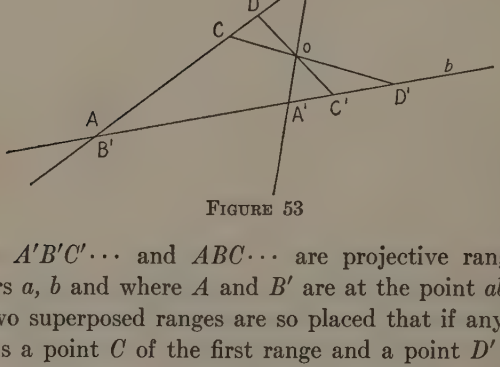

Figure 53

where $A'B'C'\cdots$ and $ABC\cdots$ are projective ranges on bearers a, b and where A and B' are at the point ab, then the two superposed ranges are so placed that if any point on a is a point C of the first range and a point D' of the second, the points C' and D also coincide; i.e., a point corresponds to the same point no matter whether it is regarded as belonging to the first or second range (Art. 39).

Similarly for two projective pencils. A section of the pencil $p'q'a'\cdots$ (Fig. 54) by a line l through O'' projected from O gives two superposed projective pencils at O so placed that if any two rays a and b' coincide so do a' and b (Art. 39). Two ranges or pencils situated in this way are said to be in involution. A point and its correspondent is called a pair. Every section or projection of an involution gives an involution.

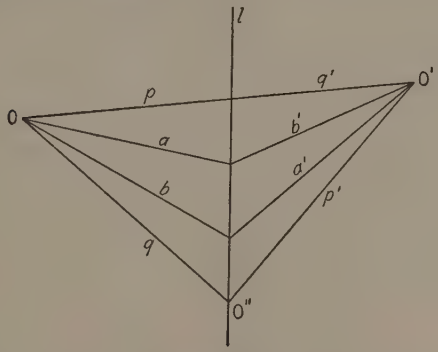

FIGURE 54

55. *Theorem:* An involution is determined by two pairs. Let A and A', B and B' be the two pairs. In the first place

$$(ABA'B') = (A'B'BA),$$

so it is not over-determined. In the next place to get the conjugate C' of a point C,

$$(AA'B'C) = (A'ABC'),$$

which determines C'; hence two pairs are sufficient to determine the correspondent of any point C.

Let O' be at infinity, then

$$(AB'OO') = (A'BO'O)$$

or

$$\frac{AO}{B'O} = \frac{BO}{A'O}, \qquad AO \cdot A'O = BO \cdot B'O = k, \text{ a constant.}$$

The point O, the conjugate of the point at infinity, is called the *centre* of the involution and the pairs of conjugate points are at such distances from O that the rectangle on the segments is constant. If the constant is positive the pairs of points are on the same side of O, if negative on opposite sides. If a point M is its own conjugate $MO^2 = k$. There are two such points equidistant from O if the constant k is positive, none if it is negative, one coincident with O if k is zero. In the last case the conjugate of every point is at O. The three kinds of involution are called hyperbolic, elliptic, and parabolic or degraded.

The double elements separate every pair of conjugate points harmonically, for if M and N are double elements,

$$(AA'MN) = (A'AMN), \qquad r = \frac{1}{r}, \qquad r = -1.$$

56. Given two pairs A and A', B and B' of an involution on a line l, to construct the centre O, the self-corresponding points if there are any, and to find other pairs.

Construction: Draw circles through AA' and BB' intersecting at R and S (Fig. 55). Let RS meet l at O, O is the centre of the involution and circles through R and S, tangent to l at points M and N, give the double points of the involution. Any circle through R and S cuts l in a pair of corresponding points.

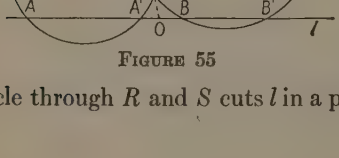

FIGURE 55

Proof:

$$OR \cdot OS = OA \cdot OA' = OB \cdot OB' = k, \text{ a constant.}$$

For the circles tangent to l at M and N

$$\overline{OM}^2 = \overline{ON}^2 = OR \cdot OS = k.$$

This is only possible when OR and OS have the same sense, i.e., both lie on the same side of l. In this case the segments AA' and BB' do not overlap.

In case AA' and BB' overlap, two circles constructed on these segments as diameters intersect at points R and S such that RS is perpendicular to l. The involution may be considered as the section by l of an involutorial pencil having either R or S as centre and such that every pair of conjugate rays is at right angles. In an involutorial pencil there are double rays which correspond to the double points on the involutorial range but the centre and point at infinity on the range do not correspond to rays of any particular interest.

57. *Exercise:*

1. Prove that in an involutorial pencil there is one rectangular pair of rays and construct them.
2. Prove that if there is more than one rectangular pair, then all pairs are rectangular.

58. *Theorem:* The three pairs of opposite sides of a complete quadrangle are cut by any transversal in three pairs of conjugate points of an involution.

Given the complete quadrangle $PQRS$ whose three pairs of opposite sides are cut by a transversal l in A and A', B and B', C and C'.

To prove that in the involution determined by A and A', B and B', C' is the conjugate to C.

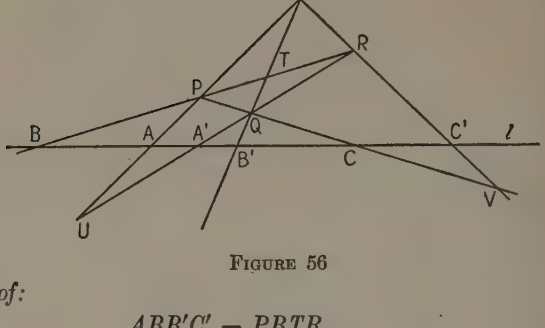

FIGURE 56

Proof:

$$ABB'C' \overline{\wedge} PBTR,$$
$$A'B'BC \overline{\wedge} RTBP \overline{\wedge} PBTR.$$

Therefore $ABB'C' \overline{\wedge} A'B'BC$ and C' is the conjugate of C in the involution determined by A and A', B and B'. The theorem can also be stated: If five sides of a complete quadrangle pass through five fixed collinear points, the sixth does likewise.

Dually: The three pairs of opposite vertices of a complete quadrilateral project from any point in three pairs of conjugate rays of an involution or if five vertices of a complete quadrilateral lie on five concurrent lines the sixth does likewise.

Exercise: Write out the proof for the dual.

59. *Desargue's Theorem:* Any transversal meets a conic and the opposite sides of an inscribed quadrangle in pairs of conjugate points of an involution.

Given a conic which cuts a line l in D and D' and a complete quadrangle inscribed in the conic whose three pairs of opposite sides cut l in A and A', B and B', C and C', points that are conjugates in an involution.

To prove that D and D' are conjugates in the involution.

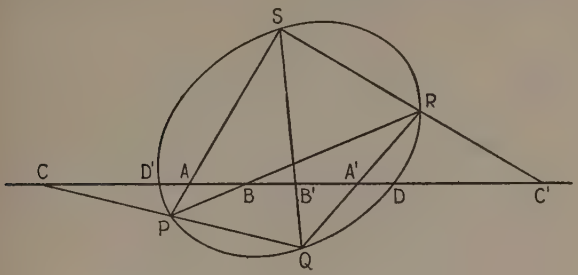

FIGURE 57

Proof: By considering pencils with vertices at Q and S it follows that (Art. 45)

$$CD'A'D \barwedge AD'C'D \barwedge C'DAD',$$

i.e., D and D' are conjugate points in the involution determined by C and C', A and A'.

Dually: The tangents from any point and the lines to the vertices of a complete quadrilateral circumscribed to a conic form pairs of conjugate rays of an involution.

Exercise: Write out the proof for the dual.

60. The theorem of Article 59 can be used to construct a conic by points, the dual to construct a conic by lines, i.e., as the envelope of its tangents. If in Figure 57 P approaches Q, there results Figure 58 which indicates how to solve the problem: Given five points $PD'SRD$ on a conic, to draw the tangent at one of them.

If in Figure 58 S and R approach coincidence, SR becomes a tangent, A and A' become coincident and give a double point of the involution in which C and C', D and D' are conjugate. If the two tangents at S and P are the

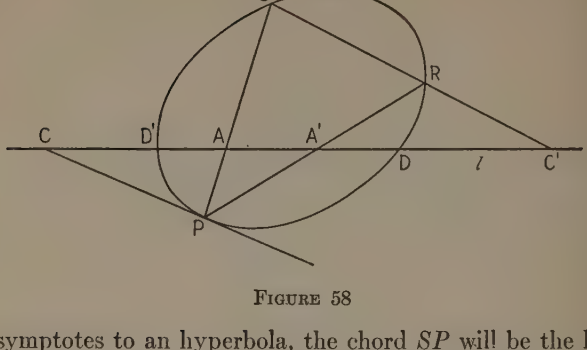

FIGURE 58

asymptotes to an hyperbola, the chord SP will be the line at infinity, hence the other double point will bisect the distance between a pair of corresponding points or DD' and CC' will have the same mid-point; hence if a line cuts a hyperbola in D and D' and its asymptotes in C and C', $DC' = CD'$.

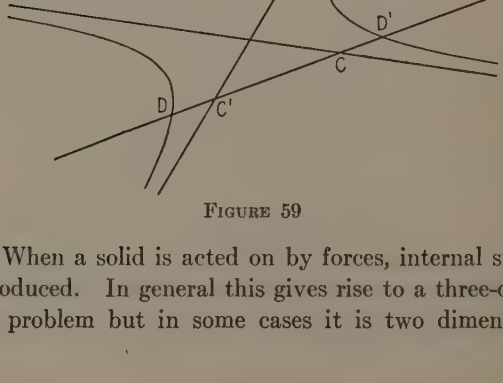

FIGURE 59

61. When a solid is acted on by forces, internal stresses are produced. In general this gives rise to a three-dimensional problem but in some cases it is two dimensional.

Thus in considering the stress on the faces of a prism, if the stresses do not vary as the altitude is changed, one may consider the stress on a prism of altitude unity and if the prism is of triangular cross-section one may think of the stresses on the faces of the prism as applied to the sides of a triangle drawn equidistant from the ends.

If the triangle is small, the force on each side may be assumed as applied at the midpoint.

The stress per unit area is the specific stress. The stress may be resolved along the surface and normal to it; the former is the shear, and the latter the tension or compression. A triangle with sides a, b, c which is in equilibrium under the action of forces A, B, C applied at the midpoint of each is shown in Figure 60. The condition for equi-

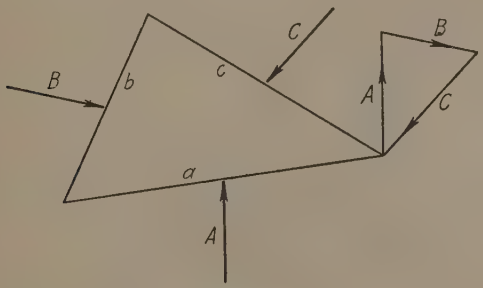

FIGURE 60

librium is that A, B, and C are concurrent and that the vectors

$$A + B + C = 0.$$

When A and B are resolved along lines parallel to a and b into the components A_a, A_b, B_a and B_b, the components A_b and B_a meet C on c so that A_a and B_b must do like-

wise; i.e., the modulus or absolute value of A_a/a is equal to the modulus of B_b/b.

If a and b are perpendicular this equation states that the specific shear on two sections at right angles has the same value. Also if $A_a = 0, B_b = 0$; i.e., if b is parallel to A, then a is parallel to B.

62. *Theorem:* A line and the direction of the stress on the line form conjugate rays of an involution.

Given a triangle with sides a, b, c and the stresses A, B, C on them such that B is parallel to a and A is parallel to b. To prove that when c rotates about (ac), c and C are conjugate rays in an involution.

At (ac) draw A and at its extremity draw a line parallel to B (Fig. 61).

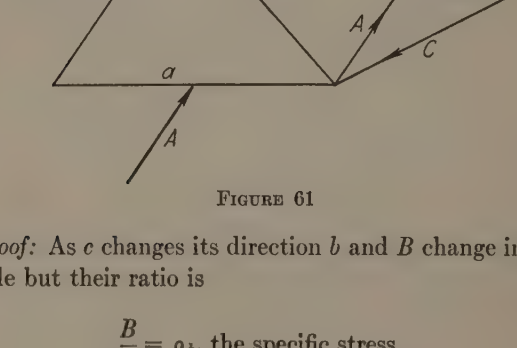

FIGURE 61

Proof: As c changes its direction b and B change in magnitude but their ratio is

$$\frac{B}{b} = \rho_b, \text{ the specific stress.}$$

Hence c and C describe ranges on b and B that are projective. Also if c_1 and C_1 are a pair of positions another

pair is obtained by interchanging c_1 and C_1. Therefore the pencils are in involutorial position.

Using a cartesian reference system with axes parallel to a and b (Fig. 62), the coordinates of the extremities of

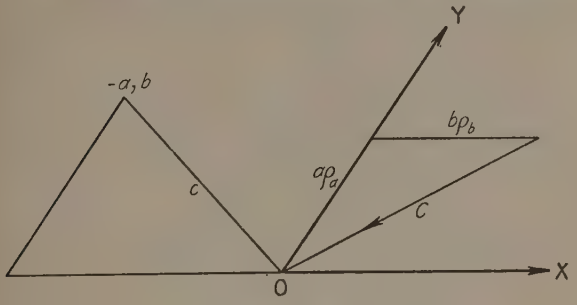

FIGURE 62

C and c are $(b\rho_b, a\rho_a)$ and $(-a, b)$. The product of the two slopes is

$$\frac{a\rho_a}{b\rho_b}\frac{-b}{a} = -\frac{\rho_a}{\rho_b},$$

which is independent of the particular direction chosen for c and therefore characterizes the involution.

Conversely if OX and OY are two conjugate directions and ρ_a and ρ_b the specific stresses on them are given, another pair of conjugate directions can be constructed as in Figure 62. In Figure 62 ρ_b and ρ_a are assumed of like sign, the two pairs of conjugate rays overlap, there are no double rays. If ρ_b and ρ_a are unlike in sign the same kind of a construction can be used but the two pairs will not overlap so that in this case there are double rays.

It is important to notice that in general the angle between a pair of conjugate rays varies and in the case of an elliptic involution has a minimum α different from zero.

In dealing with a material such as sand or earth with a friction angle φ, α must be greater than $\pi/2 - \varphi$ in order to have equilibrium.

The results may be summarized: The stresses and corresponding sections at a point form conjugate rays of an involutorial pencil. If the involution has no double rays, the stress on all sections has the same sense (tension or compression). For the normal pair the stress is perpendicular to the sections. There are no sections subjected to shear alone. There can not be more than one rectangular pair unless all pairs are rectangular. The last is the case in hydrostatics. If the involution has double rays the corresponding sections are subjected to shear alone. The double rays separate the material subjected to tension from the material subjected to compression. The rectangular pair bisects the angle of the double rays. One of these sections is subjected to tension and the other to compression.

63. *The Stress Ellipse.* Given ρ_a and ρ_b, the specific

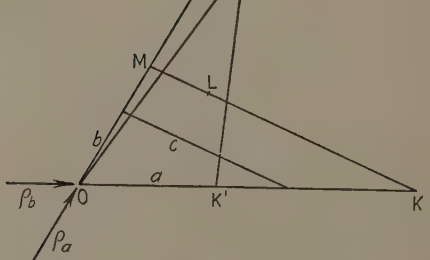

FIGURE 63

stresses on two conjugate directions a and b, to find ρ_c the specific stress on any other line.

Construction: Draw a triangle with sides a, b, c. Draw KLM parallel to c and meeting a and b at K and M (Figs. 63 and 64) and of such length that

$$KLM = KL + LM = \rho_a + \rho_b.$$

Construct the triangle $K'M'O$ equal to KMO, and take $K'L' = KL$, also draw $L'N'$ parallel to a, then $L'O = \rho_c$ in magnitude and direction.

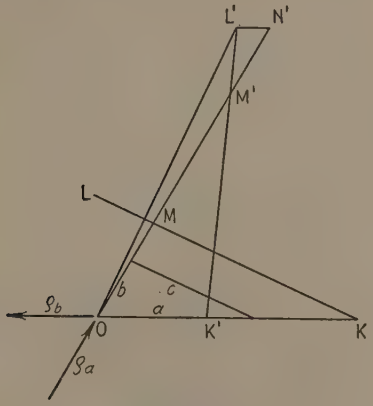

FIGURE 64

Proof:

$$ON' : K'L' = a : c, \qquad ON' = \frac{a\rho_a}{c} = \frac{A}{c},$$

$$N'L' : OK' = \rho_b : K'M', \qquad N'L' = \rho_b \frac{b}{c} = \frac{B}{c}.$$

$$\therefore L'O = \frac{C}{c} = \rho_c.$$

In either figure when the direction of c is varied, L' will be a point on a line which has two fixed points moving along given lines. The locus of L' is therefore an ellipse with the point of intersection of the fixed lines as centre.

64. *The Principal Axes of the Stress Ellipse.* Given the fixed lines a and b intersecting at O and the segment $K'M'$ which moves so that K' remains on b and M' on a, to find the principal axes of the ellipse generated by a point L' on $K'M'$ or on $K'M'$ produced.

Construction: Construct a circle through $K'OM'$ (Figs. 65 and 66). Draw a line through L' and the centre of the circle P meeting the circle in C' and O'. The distances from L' to C' and O' are the lengths of the semidiameters of the ellipse. The angle between OM' and one of the principal axes is equal to the angle $C'O'M'$.

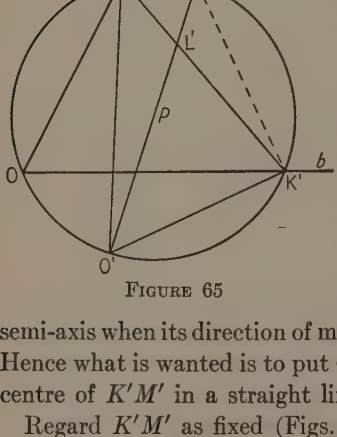

FIGURE 65

Proof: The moving point L' is at the extremity of a semi-axis when its direction of motion is perpendicular to OL' Hence what is wanted is to put O, L', and the instantaneous centre of $K'M'$ in a straight line.

Regard $K'M'$ as fixed (Figs. 65 and 66), then the locus of O with reference to $K'M'$ is a circle with O and the instantaneous centre at opposite ends of the same diameter on this circle. Hence the line from L' to the centre of the circle cuts the circle in the two points C' and O' either of which give the position of O with reference to $K'M'$. The

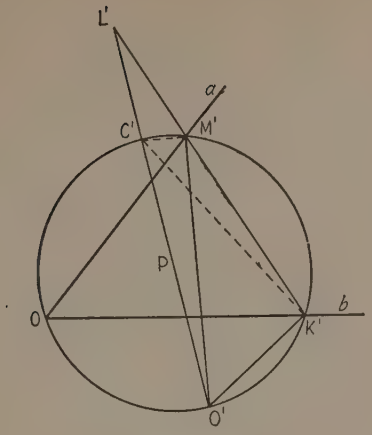

FIGURE 66

semiaxes are the two distances of L' from O', viz., $L'O'$ and $L'C'$. The angle between a semiaxis and OM' is equal to the angle $L'O'M'$.

CHAPTER IX

Pole and Polar

65. *Theorem:* The locus of points harmonically situated with respect to a point P and the two points where a variable transversal through P cuts a conic is a straight line.

Construction: Inscribe in the conic a quadrangle $ABCD$ having P as one of its diagonal points, the other diagonal points being Q and R. Draw any transversal through P meeting the conic in M and M' and the pairs of opposite sides and diagonal QR in E, E', L, L' and P. Points on the line QR satisfy the conditions of the theorem.

Proof: It has been proved (Art. 33) that the diagonal points of a complete quadrangle are harmonically situated with respect to the two points where their joining line intersects the other two sides. Consequently the pencils (Fig. 67) Q–$E'P'EP$ and R–$LP'L'P$ are both harmonic and P and P' are the double points in the involution of which E and E', M and M', etc., are conjugates (Art. 59). Therefore the locus of points harmonically situated with respect to P and the two points where any transversal cuts the conic is the line QR. The line QR is the polar line of P with respect to the given conic, P is the pole of QR. Similarly the polar line of R is the line PQ.

If A approaches B and D approaches C, DC and AB become tangents to the conic. The tangents drawn at the two points where a transversal through P cuts the conic intersect on the polar line of P. In particular the tangents to the conic from P have their point of contact

at the intersections of the conic with the polar line of P. As a transversal through P meets the conic and the polar

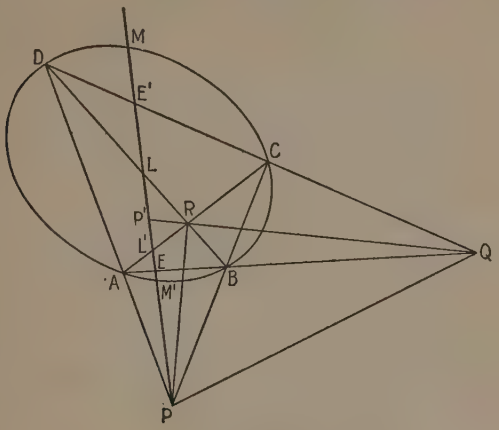

FIGURE 67

line of P in points which are harmonically situated with respect to P, it follows that when P is inside the conic, the polar line can not intersect the conic and when P is outside the polar line is the chord of contact.

If A lies on the polar line of B, then the polar line of A passes through B. Because A and B and the points of intersection of the line AB with the conic are harmonically situated. Two points so situated that each lies on the polar line of the other are conjugate points; two lines so related that each contains the pole of the other are conjugate lines, i.e., conjugate with respect to a given conic.

66. *Exercise:*

1. Find the polar line of a given point with respect to a conic, five points of the conic being given.

2. Solve the dual.
3. Find the pole of a given line with respect to the conic of Example 1.
4. Find the polar line of a given point with respect to the conic of Example 2.
5. In the last paragraph of Art. 66 it was assumed that AB intersected the conic. Prove that if A lies on the polar line of B, then the polar line of B passes through A when AB does not intersect the conic.

67. *Definition:* A triangle is self-conjugate with respect to a given base conic if each vertex is the pole of the opposite side. In constructing such a triangle, one vertex may be chosen arbitrarily, a second vertex may be chosen anywhere on the polar line of the first vertex. The polar line of the second point completes the triangle. The triangle PQR in Figure 67 is such a triangle.

Theorem: Points on a line conjugate to one another with respect to a given conic form an involution.

Given a conic and a line g (Fig. 68), to prove that the pairs of conjugate points on g form an involution.

Construction: Construct the pole G of g and inscribe a quadrangle $PQRS$ in the conic having G as a diagonal point. The points E and F are conjugate with respect to the conic. The line PQ will remain fixed but for different positions of GS different pairs of conjugate points will be obtained. To prove that E and F generate an involutorial range on g.

Proof: The pencils described by PE and QR are projective and as E and F correspond doubly, the projective ranges generated by E and F on g are in involutorial position.

If EF cuts the conic the involution is hyperbolic, otherwise elliptic unless EF is a tangent and in that case the involution is parabolic.

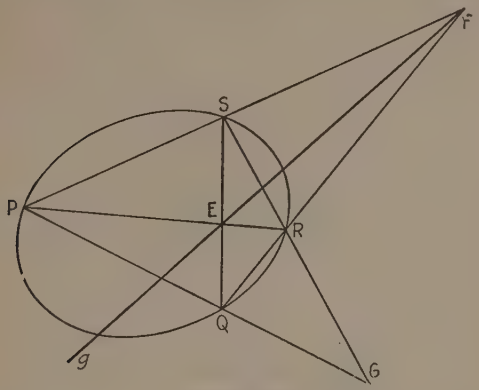

FIGURE 68

The lines from G to E and F form involutorial pencils, hence through any point there are an infinite number of lines conjugate with respect to a given conic and these form an involution. If a point describes a range its polar line describes a projective pencil, and conversely.

68. *The Centre and Diameters of a Conic.* The centre of a conic is the pole of the line at infinity. A diameter of a conic is the polar line of a point on the line at infinity. It follows immediately from the general theory that a diameter is the locus of the midpoint of a system of parallel chords and also that all diameters pass through the centre. Diameters conjugate with respect to the conic are conjugate diameters. The conjugate diameters of an ellipse form an elliptic involution, of an hyperbola a hyperbolic involution with the asymptotes as the self-corresponding rays.

In both cases there is one pair of conjugate diameters at right angles (the axes of the conic). If all are at right angles, the conic is a circle.

The parabola being tangent to the line at infinity has its centre at infinity and all its diameters are parallel lines.

Definition: If P and P' are points symmetrically situated with respect to the centre of a conic, P' is the antipole of the polar line of P, and the line is the antipolar line of P'.

69. *Exercise:*

1. Where is the antipolar line of a point on the conic?
2. For what line do pole and antipole coincide?
3. Find the centre of a conic which is given by five points.
4. The diagonals of any parallelogram circumscribed to a conic are conjugate with respect to the conic.

70. *The Kern.* In structural work it is important to know over what part of the cross section of a column a normal force may be applied without producing tension over any part of the cross section. The area over which this is the case is the kern for the given cross section.

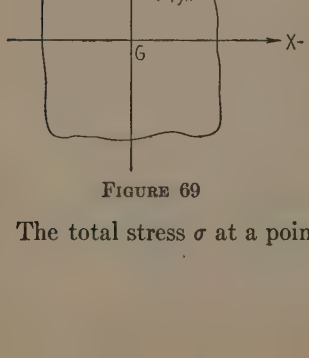

FIGURE 69

Figure 69 represents the cross section of a column of area A, acted on by a force F, normal to the section at (x_1, y_1). The origin is at the centre of gravity of the section and the axes of reference are the principal axes.

The total stress σ at a point (x, y) is made up as follows:

$\frac{F}{A} = \sigma_3 =$ stress considered as due to uniform force over A.

A part σ_1 due to the moment of F about OX such that

$$Fy_1 = \sum yky\Delta A = kI_{OX}, \qquad \text{but} \qquad ky = \sigma_1, \qquad k = \frac{\sigma_1}{y}$$

so that

$$Fy_1 = \frac{\sigma_1 I_{OX}}{y} \qquad \text{or} \qquad \sigma_1 = \frac{Fy_1 y}{I_{OX}}.$$

Similarly due to the moment of F about OY there is a stress σ_2 such that

$$\sigma_2 = \frac{Fx_1 x}{I_{OY}}.$$

Hence the total stress

$$\sigma = \sigma_1 + \sigma_2 + \sigma_3$$

$$= \frac{F}{A} + \frac{Fy_1 y}{I_{OX}} + \frac{Fx_1 x}{I_{OY}} = \frac{F}{A}\left(1 + \frac{yy_1}{r_{OX}^2} + \frac{xx_1}{r_{OY}^2}\right).$$

For $\sigma = 0$,

$$1 + \frac{yy_1}{r_{OX}^2} + \frac{xx_1}{r_{OY}^2} = 0.$$

This is the equation of the antipolar line of (x_1, y_1) with respect to the ellipse

$$\frac{x^2}{r_{OY}^2} + \frac{y^2}{r_{OX}^2} = 1.$$

All that is necessary to find the kern is to find the locus of the antipole (x_1, y_1) when the antipolar line is the envelope of the curve bounding A.

71. *Exercise:*

1. Find the kern for a circular section of radius a.
2. Find the kern for a rectangular cross section.
3. Find the kern if the cross section is a semicircle.

4. Find the kern for the area bounded by the parabolas $y = 1 - x^2$ and $y = x^2 - 1$.

72. The idea of the antipole also enters in hydrostatics. Let A be a vertical area (Fig. 70) immersed in water with the centre of gravity G of the area at a depth p below the surface of the water. The principal axes of A are the axes of reference and the x axis makes an angle α with the surface of the water. It is required to find the coordinates (x_c, y_c) of the centre of pressure.

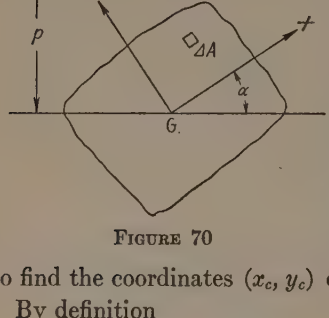

FIGURE 70

By definition

$$x_c = \frac{\Sigma x \Delta A(p - x\sin\alpha - y\cos\alpha)}{\Sigma \Delta A(p - x\sin\alpha - y\cos\alpha)},$$

$$y_c = \frac{\Sigma y \Delta A(p - x\sin\alpha - y\cos\alpha)}{\Sigma \Delta A(p - x\sin\alpha - y\cos\alpha)}.$$

But

$$\Sigma\Delta A\, p - \sin\alpha \Sigma x \Delta A - \cos\alpha \Sigma y \Delta A = pA,$$
$$p\Sigma x\Delta A - \sin\alpha \Sigma x^2 \Delta A - \cos\alpha \Sigma xy\Delta A = -\sin\alpha I_{OY},$$
$$p\Sigma y\Delta A - \sin\alpha \Sigma xy\Delta A - \cos\alpha \Sigma y^2 \Delta A = -\cos\alpha I_{OX}.$$

Hence

$$x_c = -\frac{r_{OY}^2 \sin\alpha}{p}, \qquad y_c = -\frac{r_{OX}^2 \cos\alpha}{p}.$$

Now the equation of the line at the water surface is

$$\frac{x\sin\alpha}{p} + \frac{y\cos\alpha}{p} = 1.$$

And this is the antipolar line of (x_c, y_c) with respect to the

ellipse

$$\frac{x^2}{r_{OY}^2} + \frac{y^2}{r_{OX}^2} = 1.$$

That is, the centre of pressure is the antipole of the surface line with respect to the ellipse

$$\frac{x^2}{r_{OY}^2} + \frac{y^2}{r_{OX}^2} = 1.$$

73. *Polar Reciprocal Figures.* Polar reciprocal figures are figures so related that to a line in one corresponds a point in the other and conversely. To obtain such figures it is first of all necessary to establish a correspondence between the points and lines of a plane. One method of establishing such a correspondence has already been carried out by means of an auxiliary conic, K.

To get the reciprocal of C with respect to K, one may consider C as a line locus, i.e., as the envelope of a family of lines, the reciprocal curve C' will then be generated as a point locus by the motion of the pole of the moving tangent which generates C. As two tangents to C approach coincidence their point of intersection approaches the point of contact. This point corresponds to the line which is obtained by joining the poles of the two tangents and when these approach coincidence, their joining line is a tangent to C'. Hence the reciprocal curve is the same no matter whether C is considered as the envelope of a family of lines or the locus of a moving point.

To the n points of intersection of a line p with C there corresponds n tangents from its pole P to C' and to the m tangents from a point Q to C there correspond m intersections of the polar line of Q with C', i.e., the order of a curve is equal to the class of its dual or polar reciprocal.

A conic section is of the second order and second class. Hence the polar reciprocal of a conic section is another conic section.

74. The correspondence just discussed may also be established without giving K, and in its place giving three lines and their poles. Three lines and their poles determine the conic which may or may not be real. This might be expected, for giving a line and assigning a definite pole to it imposes two conditions as any coordinates (x, y) may be chosen for the pole. Assigning poles to three lines imposes six conditions, one more than a conic can satisfy, so it is to be expected that a condition must be imposed on the situation of the three poles. As a preliminary theorem the following will be proved.

Theorem: If two pairs of opposite vertices of a complete

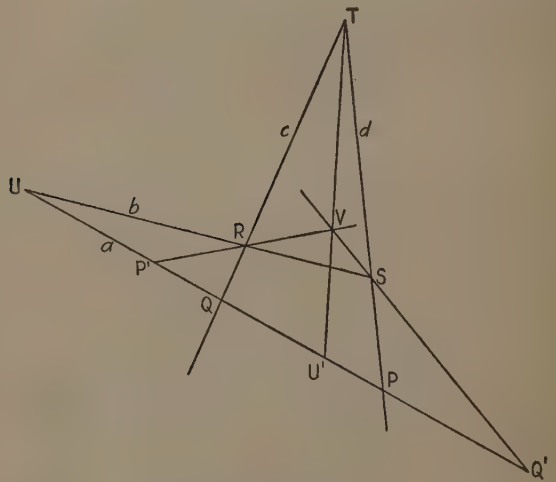

FIGURE 71

quadrilateral are conjugate with respect to a given conic the third pair is likewise.

Given the complete quadrilateral *abcd* (Fig. 71) with the pairs of opposite vertices P, R and Q, S conjugate with respect to a conic K, to prove the remaining pair T and U are also conjugate with respect to K.

Proof: Suppose the polar of P is $P'R$ and the polar of Q is $Q'S$. Then P, P' and Q, Q' determine an involution and by considering the quadrangle $SVRT$ it appears that U and U' are conjugates in this involution. But V, the intersection of the polar lines of P and Q, is the pole of PQ, and U is the pole of $U'T$. Hence U and T are conjugate.

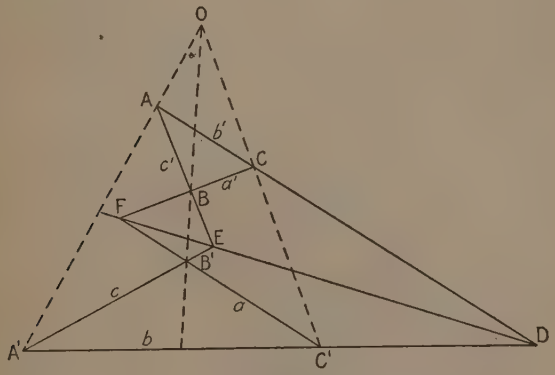

FIGURE 72

75. *Theorem:* The triangles formed by three points and by the polar lines of the three points are in plane homology.

Given the points A, B, C and their polar lines a, b, c forming the triangle $A'B'C'$, to prove that ABC and $A'B'C'$ are in plane homology (Fig. 72).

Construction: Let b and b' meet in D and a and a' in F and let c cut DF in E; it is necessary to prove that c' also passes through E. Note that A', B', C' are the poles of a', b', c'.

Proof: In the complete quadrilateral formed by a, b, c, and DF, A' and F, B' and D are conjugates; hence C' and E are also conjugates and therefore c' passes through E and the triangles are in plane homology.

76. *The Auxiliary Conic.* Referring to Figure 72, it is noted that the points $(a'b)$ and A' are conjugate, also $(c'b)$ and C'. This determines an involution on b and the double points of this involution are points on the auxiliary conic. If the involution is elliptic B is inside the conic, if the conic is real, and in that case there will be real double points in the corresponding involutions on a' and c'.

So far as determining the polar line of any point or the pole of any line the two triangles furnish all the data needed, irrespective of whether the conic is real or not.

It may be noted that in Figure 72, E is the pole of CC', F of AA', and D of BB', so the centre of homology is the pole of the axis of homology.

77. *Exercise:*

1. If two lines and their poles are given, show how to find the pole of any line through the point of intersection of the two lines.
2. Show how to find the pole of any line and also the polar line of any point when three lines and their poles are given.

78. A very simple illustration of a polarity is furnished by the motion of a plate in its plane. Supposing the plate initially at rest, an impulsive force applied to the plate sets it in motion with a certain point as instantaneous

centre. This point depends only on the line of action of the impulse, the point is the pole of the line. To every line in the plane there will be a definite pole. If the line of action of the impulsive force passes through the centre of gravity of the plate, the resulting motion is a translation. The centre of gravity is the pole of the line at infinity. The auxiliary conic is imaginary because the effect of the impulse is never such as to produce a rotation about a point on its line of action, hence there is no line which contains its own pole. The auxiliary conic has no real tangents.

It is readily shown that the instantaneous centre is the antipole of the line of action of the impulsive force with respect to a circle whose centre is at the centre of gravity of the plate and whose radius is the radius of gyration of the plate for the centroid.

CHAPTER X

The Null System

79. A correspondence in which to any plane corresponds a pole situated in the plane and all planes through a point have their poles in the polar plane of the point is called a null system.

Let A, B, C, D be four points in a plane δ, no three being collinear. Draw planes α, β, γ through AD, BD and CD (Fig. 73). Let their lines of intersection be $\alpha\gamma$, $\alpha\beta$, $\beta\gamma$ and draw a plane σ meeting δ in l and cutting $\alpha\gamma$, $\alpha\beta$, $\beta\gamma$ in P, Q, R. Let l meet the sides of the complete quadrangle $ABCD$ in X, Y, Z, X', Y', Z', three pairs of points of an involution. The planes RBC, PAC and QAB intersect σ in the lines $X'R$, $Y'P$ and $Z'Q$. They consequently pass through a common point S which with P, Q and R are the vertices of a complete quadrangle. The theorem as stated by Möbius is: "If the faces of a tetrahedron $ABCS$ pass through the vertices of another tetrahedron $PQRD$ and if three faces of the latter pass through three vertices of the former, then the fourth face of the second will pass through the fourth vertex of the first."

The point S has been found when the plane σ was given, conversely when S is given the planes SBC, SAC, SBA determine the points R, P, Q and hence the plane σ. S is the pole of σ and σ is the polar plane of S.

If σ rotates about the line PR, the pole S will move along the line of intersection of the planes PAC and RBC and when Q falls at D, S will coincide with C; similarly A is the pole of α, B of β. If σ passes through BC, S falls at R.

The poles of RBC, PAC and QAB are R, P and Q. If σ initially contains CB and then rotates about CB till it

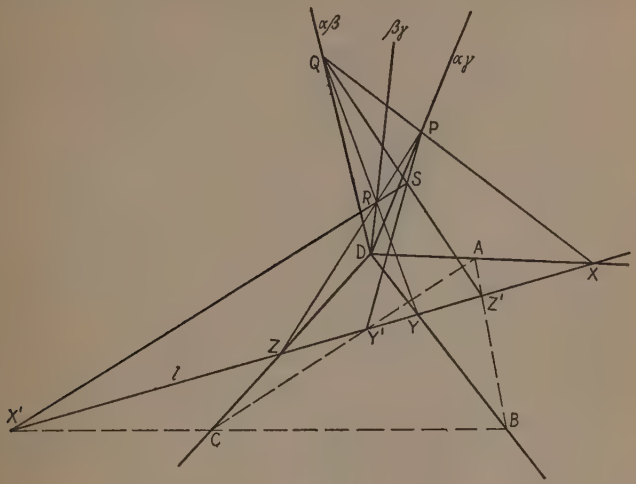

FIGURE 73

coincides with δ, the pole falls at D. In case σ passes through C, the planes RBC and PAC intersect in CRP, a line in γ, so the poles of all planes through a point lie on the line of intersection of the given plane with the polar plane of the point. It also follows that the polar planes of all points in a plane α pass through the pole of α. For if a point B in α has a polar plane β, then the pole of α must lie on $\alpha\beta$.

80. Let α and β be two planes with poles A and B. Any plane through AB has its pole at the intersection with $\alpha\beta$ and any plane through $\alpha\beta$ has its pole on AB. Two lines each of which is the locus of poles of planes passing through

the other are reciprocal. If a straight line passes through A, its reciprocal lies in the polar plane of A. A line which lies in the polar plane of A and passes through A is its own reciprocal or self-reciprocal. A system of lines which are self-reciprocal form a *linear complex*. The lines are the rays of the complex. Each ray of a complex which meets one line also meets its reciprocal. For if r and r' are reciprocal lines, the polar plane of any point A on r is the plane Ar'. So if a line t of the complex meets r at A, t will lie in the plane Ar'. Conversely a line which meets two reciprocal lines r and r' belongs to the complex.

All parallel planes have a common line r' at infinity. Therefore their poles all lie on r the reciprocal of r'. Changing the direction of the pencil of parallel planes, the line reciprocal to their line at infinity maintains the same direction as it passes through I, the pole of the plane at infinity. Lines r whose reciprocals lie at infinity are called diameters of the complex. The diameter perpendicular to the planes whose poles it contains is called the central axis of the complex.

The line which passes through the points at infinity of reciprocal lines is a ray of the complex and therefore passes through I. Hence two reciprocal straight lines and the central axis are parallel to the same plane. Two reciprocal straight lines projected parallel to the central axis on a plane give two parallel straight lines.

Take the Z axis as the central axis and make the projection on a plane perpendicular to Z; this is the orthographic plane. Then as any plane through the pole of another has its pole on their line of intersection, and as every point on the Z axis is the pole of the plane parallel to XOY through that point, it follows that the plane

$$ax + by + cz + d = 0$$

has its pole at

$$z_1 = -\frac{d}{c} \text{ and } \frac{x_1}{y_1} = -\frac{b}{a}, \text{ i.e., } b = -kx_1 \text{ and } a = ky_1.$$

The equation of the plane can be written

$$kxy_1 - kyx_1 + cz - cz_1 = 0,$$

or

$$xy_1 - yx_1 + k'(z - z_1) = 0,$$

where k' is a constant, as the poles of parallel planes lie on a diameter.

81. If $A, B, C, D\cdots$ are vertices of a polyhedron, their polar planes $\alpha, \beta, \gamma, \delta\cdots$ are faces of a second polyhedron; the two are reciprocal. Their projections on the orthographic plane give figures with reciprocal properties. To each side corresponds a parallel side, to a point corresponds a closed polygon. Three sides at least meet at a vertex. If one polyhedron has a vertex at infinity, the other has a face perpendicular to the orthographic plane, so if one of the projections has a vertex at infinity, the projection of the other contains a polygon, whose sides lie in the same straight line. This is precisely the kind of reciprocal figures which arise in graphical statics.

82. A system of forces in space defines a null system. The different pairs of reciprocal lines are the two lines along which the two skew forces to which the system can be reduced may act. A plane containing one force has its pole at the point where it is cut by the other. If OZ is taken as central axis, the system reduces to a force R along OZ and a couple of moment $H = -Fp$, whose vector is parallel to OZ. The couple H can be replaced by a force F, parallel to Ox, and lying in the XZ plane,

together with an equal and parallel force but of opposite sense at a distance p from the XZ plane.

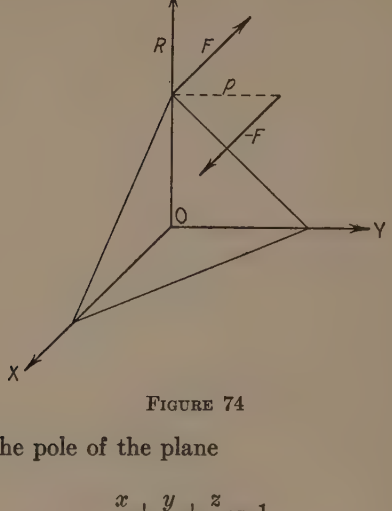

FIGURE 74

To find the pole of the plane

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1,$$

first determine p so that the resultant of R and F lies in the plane, the point where the force $-F$ pierces the plane is the pole. The resultant of F and R lies in the plane if

$$\frac{F}{R}=\frac{a}{c}. \quad \text{Hence } p=y_1=-\frac{cH}{aR}.$$

Now $z_1=c$, and therefore

$$\frac{x_1}{a}+\frac{y_1}{b}=0, \qquad x_1=-\frac{a}{b}y_1=\frac{cH}{bR}.$$

These values are introduced in the equation of the plane by expressing a, b, c in terms of x_1, y_1, z_1, which gives

$$a = -\frac{z_1 H}{y_1 R}, \qquad b = \frac{z_1 H}{x_1 R}, \qquad c = z_1.$$

Substitution in the equation of the plane and simplification gives

$$xy_1 - yx_1 - \frac{H}{R}(z - z_1) = 0,$$

as in Article 80.

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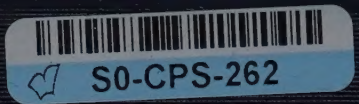
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